2A-orbifold construction and the baby-monster vertex operator superalgebra

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Abstract

In this article we prove that the full automorphism group of the baby-monster vertex operator superalgebra constructed by Höhn is isomorphic to $2\times\mathbb{B}$, where $\mathbb{B}$ is the baby-monster sporadic finite simple group, and determine irreducible modules for the baby-monster vertex operator algebra. Our result has many corollaries. In particular, we can prove that the $\mathbb{Z}_2$-orbifold construction with respect to a 2A-involution of the Monster applied to the moonshine vertex operator algebra $V^\natural$ yields $V^\natural$ itself again.

1 Introduction

The famous moonshine vertex operator algebra $V^\natural$ constructed by Frenkel-Lepowsky-Muerman [FLM] is the first example of the $\mathbb{Z}_2$-orbifold construction of a holomorphic vertex operator algebra (VOA). Let us explain a $\mathbb{Z}_2$-orbifold construction briefly. Let V be a holomorphic vertex operator algebra and σ an involutive automorphism on V. Then the fixed point subalgebra $V^{\langle\sigma\rangle}$ is a simple vertex operator algebra. It is shown in [DLM1] that there is a unique irreducible σ-twisted V-module M and we have a decomposition $M=M^0\oplus M^1$ into a direct sum of irreducible $V^{\langle\sigma\rangle}$-modules such that M^0 has an integral top weight. Then a $\mathbb{Z}_2$-orbifold construction with respect to $\sigma\in\mathrm{Aut}(V)$ is to construct a $\mathbb{Z}_2$-graded extension $W=V^{\langle\sigma\rangle}\oplus M^0$ of the fixed point subalgebra $V^{\langle\sigma\rangle}$ which is expected to be a holomorphic vertex operator algebra.

In FLM's construction, we take V to be the lattice vertex operator algebra V_Λ associated to the Leech lattice Λ and the involution σ is a natural lifting $\theta\in\mathrm{Aut}(V_\Lambda)$ of the (-1)-isometry on Λ. Denote by $V_\Lambda=V_\Lambda^+\oplus V_\Lambda^-$ the eigenspace decomposition such that θ acts on $V_\Lambda^\pm$ as ±1. Let V_Λ^T be the unique irreducible θ-twisted V_Λ-module. Then there is a decomposition $V_\Lambda^T=(V_\Lambda^T)^+\oplus(V_\Lambda^T)^-$ such that the top weight of $(V_\Lambda^T)^+$ is integral. Then

the moonshine vertex operator algebra is defined by $V^\natural := V_\Lambda^+ \oplus (V_\Lambda^T)^+$ and it is proved in [FLM] that $V^\natural$ forms a $\mathbb{Z}_2$-graded extension of V_Λ^+. It is also proved in [FLM] that the full automorphism group of the moonshine vertex operator algebra is the Monster sporadic finite simple group $\mathbb{M}$ by using Griess' result [G].

In the Monster, there are two conjugacy classes of involutions, the 2A-conjugacy class and the 2B-conjugacy class (cf. [ATLAS]). One can explicitly see the action of a 2B-involution on $V^\natural$ by FLM's construction. But it is not clear to see the action of a 2A-involution on $V^\natural$ before Miyamoto. In [M1], Miyamoto opened a way to study the action of 2A-involutions of the Monster on the moonshine VOA by using a sub VOA isomorphic to the unitary Virasoro VOA $L(1/2,0)$. Let us recall the definition of Miyamoto involutions. Let V be a simple VOA and $e \in V_2$ be a vector such that e generates a sub VOA isomorphic to $L(1/2,0)$. Such a vector e is called conformal vector with central charge $1/2$. Since V as a $\mathrm{Vir}(e) \simeq L(1/2,0)$-module is completely reducible, we have a decomposition

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16),$$

where $V_e(h)$ denotes a sum of all irreducible $\mathrm{Vir}(e)$-submodules isomorphic to $L(1/2,h)$, $h = 0, 1/2, 1/16$. Then one can define a linear isomorphism τ_e on V by

$$\tau_e := \ 1 \quad \text{on} \quad V_e(0) \oplus V_e(1/2), \quad -1 \quad \text{on} \quad V_e(1/16).$$

Then it is proved in [M1] that τ_e defines an involution of a VOA V if $V_e(1/16) \neq 0$. This involution is often called the (first) Miyamoto involution. If $V_e(1/16) = 0$, then one can define another automorphism on V by

$$\sigma_e := \ 1 \quad \text{on} \quad V_e(0), \quad -1 \quad \text{on} \quad V_e(1/2).$$

This involution is also called the (second) Miyamoto involution. It is shown in [C] and [M1] that in the moonshine VOA every Miyamoto involution τ_e defines a 2A-involution of the Monster and the correspondence between conformal vectors and 2A-involutions is one-to-one. Therefore, in the study of 2A-involutions, it is very important to study conformal vectors with central charge $1/2$. Along this idea, C.H. Lam, H. Yamada and the author obtained an interesting achievement on 2A-involutions of the Monster in [LYY].

The main purpose of this paper is to study the $\mathbb{Z}_2$-orbifold construction of $V^\natural$ with respect to the Miyamoto involution and to prove that the 2A-orbifold construction applied to $V^\natural$ yields $V^\natural$ itself again. Since a 2A-involution of the Monster is uniquely determined by a conformal vector e of $V^\natural$ with central charge $1/2$, we first study the commutant subalgebra of $\mathrm{Vir}(e)$. For a simple VOA V and a conformal vector e of V with central charge $1/2$, set the space of highest weight vectors by $T_e(h) := \{v \in V \mid e_{(1)}v\}$ for $h = 0, 1/2, 1/16$. Then we have decompositions $V_e(h) = L(1/2,h) \otimes T_e(h)$ and the

commutant subalgebra $T_e(0)$ acts on $T_e(h)$ for $h = 0, 1/2, 1/16$. Since $L(1/2, 0)$ has a $\mathbb{Z}_2$-graded extension $L(1/2, 0) \oplus L(1/2, 1/2)$, we can introduce a vertex operator superalgebra (SVOA) structure on $T_e(0) \oplus T_e(1/2)$ (Theorem 3.6) and its $\mathbb{Z}_2$-twisted module structure on $T_e(1/16)$ (Theorem 3.8). It is easy to see that the one point stabilizer $C_{\mathrm{Aut}(V)}(e)$ of a conformal vector e naturally acts on the space of highest weight vectors $T_e(h)$. If we take $V = V^\natural$, then $C_{\mathrm{Aut}(V^\natural)}(e)$ is isomorphic to the 2-fold central extension $\langle \tau_e \rangle \cdot \mathbb{B}$ of the baby-monster sporadic finite simple group $\mathbb{B}$. Therefore, the SVOA $T_e^\natural(0) \oplus T_e^\natural(1/2)$, where we have set $V_e^\natural(h) = L(1/2, h) \otimes T_e^\natural(h)$ for $h = 0, 1/2, 1/16$, affords a natural action of $\mathbb{B}$. Motivated by this fact, Höhn first studied this SVOA in [Hö1] and he called it the baby-monster SVOA. Following him, we write $V\!B^0 := T_e^\natural(0)$, $V\!B^1 := T_e^\natural(1/2)$ and $V\!B := T_e^\natural(0) \oplus T_e^\natural(1/2)$. It is proved in [Hö2] that the full automorphism group of the even part $V\!B^0$ of $V\!B$ is exactly isomorphic to the baby-monster $\mathbb{B}$. In this paper, we give a quite different proof of $\mathrm{Aut}(V\!B^0) \simeq \mathbb{B}$ based on a theory of simple current extensions.

In my recent work [Y1] [Y2], a theory of simple current extensions of vertex operator algebras was developed and many useful results were obtained. Using this theory, we determine the automorphism group of the commutant subalgebra $T_e(0)$ as follows:

Theorem 1. *Let V be a holomorphic VOA and $e \in V$ a conformal vector with central charge 1/2. Suppose the following:*
(a) $V_e(h) \neq 0$ for $h = 0, 1/2, 1/16$,
(b) $V_e(0)$ and $T_e(0)$ are rational C_2-cofinite VOAs of CFT-type,
(c) $V_e(1/16)$ is a simple current $V^{\langle \tau_e \rangle}$-module,
(d) $T_e(1/2)$ is a simple current $T_e(0)$-module,
(e) $C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle$ is a simple group or an odd group.
Then
(i) $\mathrm{Aut}(T_e(0)) = C_{\mathrm{Aut}(V)}(e)/\langle \tau_e \rangle$.
(ii) The irreducible $T_e(0)$-modules are given by $T_e(0)$, $T_e(1/2)$ and $T_e(1/16)$.
(iii) The τ_e-orbifold construction applied to V yields V itself again.

The assumptions (c) and (d) in the theorem above seem to be rather restrictive. However, we prove that all the assumptions above hold if V is the moonshine VOA. We also present a refinement of Miyamoto's reconstruction of the moonshine VOA [M5]. Our refinement enable us to prove not only that the baby-monster SVOA $V\!B$ satisfies all the assumptions above but also that we can construct the baby-monster SVOA $V\!B$ without reference to $V^\natural$. The main theorem of this paper is

Theorem 2. *Let $V\!B = V\!B^0 \oplus V\!B^1$ the simple SVOA obtained from $V^\natural$.*
(1) $\mathrm{Aut}(V\!B^0) = \mathbb{B}$ *and* $\mathrm{Aut}(V\!B) = 2 \times \mathbb{B}$.
(2) There are exactly three irreducible $V\!B^0$-modules, $V\!B^0$, $V\!B^1$ and $V\!B_T := T_e^\natural(1/16)$.

(3) The fusion rules for $V\!B^0$-modules are as follows:

$$V\!B^1 \times V\!B^1 = V\!B^0, \quad V\!B^1 \times V\!B_T = V\!B_T, \quad V\!B_T \times V\!B_T = V\!B^0 + V\!B^1.$$

This theorem has many corollaries:

Corollary 1. *The irreducible 2A-twisted $V^\natural$-module has a shape*

$$L(1/2,1/2) \otimes V\!B^0 \oplus L(1/2,0) \otimes V\!B^1 \oplus L(1/2,1/16) \otimes V\!B_T.$$

Corollary 2. *For any conformal vector $e \in V^\natural$ with central charge 1/2, there is no $\rho \in \mathrm{Aut}(V^\natural)$ such that $\rho(V_e^\natural(h)) = V_e^\natural(h)$ for $h = 0, 1/2, 1/16$ and $\rho|_{(V^\natural)^{\langle \tau_e \rangle}} = \sigma_e$.*

Corollary 3. *The 2A-orbifold construction applied to the moonshine VOA $V^\natural$ yields $V^\natural$ itself again.*

At the end of this paper, we give characters of $V\!B^0$-modules and their modular transformation laws. Surprisingly, we find that the fusion algebra and the modular transformation laws for the baby-monster VOA is canonically isomorphic to those of the Ising model $L(1/2,0)$.

2 Simple current extension

Let V be a simple vertex operator algebra (VOA). We recall a definition of a fusion product of V-modules.

Definition 2.1. Let M^1, M^2 be V-modules. A *fusion product* for the ordered pair (M^1, M^2) is a pair $(M^1 \boxtimes_V M^2, F(\cdot,z))$ consisting of a V-module $M^1 \boxtimes_V M^2$ and a V-intertwining operator $F(\cdot,z)$ of type $M^1 \times M^2 \to M^1 \boxtimes_V M^2$ satisfying the following universal property: For any V-module W and any V-intertwining operator $I(\cdot,z)$ of type $M^1 \times M^2 \to W$, there exists a unique V-homomorphism ψ from $M^1 \boxtimes_V M^2$ to W such that $I(\cdot,z) = \psi F(\cdot,z)$. We usually denote the pair $(M^1 \boxtimes_V M^2, F(\cdot,z))$ simply by $M^1 \boxtimes_V M^2$.

A theory of fusion products has been greatly developed by Huang-Lepowsky [HL1]-[HL4] and Huang [H1]-[H4] (see also [DLM2] [Li3]), and it is proved that if V is rational then a fusion product of any two V-modules always exists (cf. [HL3] [HL4] [Li3]) and if V is also C_2-cofinite and of CFT-type then the fusion product satisfies the associativity (cf. [H1] [H4] [DLM2]). Therefore, the theory of fusion products is a powerful tool to study a rational C_2-cofinite vertex operator algebra of CFT-type. Among modules for such a vertex operator algebra, simple current modules have a special importance.

Definition 2.2. An irreducible V-module U is called a *simple current* if it satisfies: For any irreducible V-module W, the fusion product $U \boxtimes_V W$ is also irreducible.

In this paper we mainly consider the following extensions of vertex operator algebras.

Definition 2.3. Let V^0 be a simple rational VOA and D a finite abelian group. Let $\{V^\alpha \mid \alpha \in D\}$ be a set of inequivalent irreducible V^0-modules. A *D-graded extension V_D of V^0* is a simple VOA $V_D = \oplus_{\alpha\in D} V^\alpha$ which extends the VOA structure of V^0 with the grading structure $Y_{V_D}(x^\alpha, z)x^\beta \in V^{\alpha+\beta}((z))$ for any $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$. A D-graded extension V_D is called a *D-graded simple current extension of V^0* if all V^α, $\alpha \in D$, are simple current V^0-modules. In the case of $D = \mathbb{Z}_2 = \{0, 1\}$ and $V_D = V^0 \oplus V^1$ is a simple vertex operator superalgebra with even part V^0 and odd part V^1, we call V_D a *simple current super-extension of V^0* if V^1 is a simple current V^0-module.

Remark 2.4. Let D^* be the dual group of D. By definition, there is a natural action of D^* on V_D defined as $\chi|_{V^\alpha} = \chi(\alpha)\cdot \mathrm{id}_{V^\alpha}$ for $\alpha \in D$ and $\chi \in D^*$.

We have the following uniqueness property of a simple current extension.

Lemma 2.5. *([DM2]) (i) Let $V_D = \oplus_{\alpha\in D} V^\alpha$ be a D-graded extension of V^0. If the space of V^0-intertwining operators of type $V^\alpha \times V^\beta \to V^{\alpha+\beta}$ is one-dimensional for all $\alpha, \beta \in D$, then the VOA structure on V_D is unique over $\mathbb{C}$. In particular, if V_D is a D-graded simple current extension of V^0, then its VOA structure is unique over $\mathbb{C}$.*
(ii) The SVOA structure on a simple current super-extension is unique over $\mathbb{C}$.

In general, it is a difficult problem to determine whether a given module is a simple current or not. However, the following lemma provides us a simple criterion.

Lemma 2.6. *([Y2]) Let V be a simple rational C_2-cofinite VOA of CFT-type and U a V-module. If there is a V-module W such that the fusion rule $U \boxtimes_V W = V$ holds in the fusion algebra for V, then U (and also W) is a simple current V-module.*

Proof: By the assumption, we can use the results in [H4] so that the fusion algebra for V is a commutative associative algebra over $\mathbb{N}$ with the unit element V. Let X be an irreducible V-module. Then $U \boxtimes_V X$ is also a V-module and is a direct sum of irreducible V-modules. Let $U \boxtimes_V X = \oplus_{i\in I} T^i$ be a decomposition into a direct sum of irreducible V-modules. First, we show that $W \boxtimes_V T^i \neq 0$ for all $i \in I$. Assume that there is an $i_0 \in I$ such that $W \boxtimes_V T^{i_0} = 0$. Then by multiplying U in the fusion algebra we obtain $0 = U \boxtimes_V (W \boxtimes_V T^{i_0}) = (U \boxtimes_V W) \boxtimes_V T^{i_0} = V \boxtimes_V T^{i_0} = T^{i_0}$, a contradiction. Therefore, $W \boxtimes_V T^i \neq 0$ for all $i \in I$. Then by multiplying W to $U \boxtimes_V X$ we get $X = V \boxtimes_V X = (W \boxtimes_V U) \boxtimes_V X = W \boxtimes_V (U \boxtimes_V X) = W \boxtimes_V \oplus_{i\in I} T^i = \oplus_{i\in I} W \boxtimes_V T^i$.

Therefore, the cardinality of the index set I is 1 and hence $U \boxtimes_V X$ is an irreducible V-module. Thus U is a simple current V-module. ∎

The representation theory of simple current extensions is studied in many papers (cf. [DLM3] [L1] [SY] [Y1] [Y2]) and it is shown in [L1] [Y1] [Y2] that every simple current extension of a simple rational C_2-cofinite VOA of CFT-type is also rational and C_2-cofinite. We review some results from [Y1] and [Y2] which we will need later.

Let $V_D = \oplus_{\alpha\in D} V^\alpha$ be a D-graded simple current extension of a simple rational C_2-cofinite CFT-type VOA V^0. Since the fusion algebra for V^0 is associative, we can adopt the following definition.

Definition 2.7. Let W be an irreducible V^0-module W. A subset $D_W := \{\alpha \in D \mid V^\alpha \boxtimes_V W \simeq W\}$ forms a subgroup of D. We call D_W the *stabilizer* of W.

Lemma 2.8. *([SY]) Let M be a V_D-module and W an irreducible V^0-submodule of M. Then $V^\alpha \cdot W$ is also a non-trivial irreducible V^0-submodule of M, where $V^\alpha \cdot W$ denotes a linear space spanned by elements $a_{(n)}w = \mathrm{Res}_z z^n Y_M(a, z)w$ with $a \in V^\alpha$, $w \in W$ and $n \in \mathbb{Z}$. The stabilizer D_W is determined independently of a choice of an irreducible V^0-submodule W if M is an irreducible V_D-module.*

By the lemma above, we introduce the following notion.

Definition 2.9. An irreducible V_D-module M is called *D-stable* if $D_W = 0$ for some irreducible V^0-submodule W of M.

Among V_D-modules, D-stable modules enjoy nice properties.

Proposition 2.10. *([SY] [Y1]) Let M be an irreducible D-stable V_D-module. Then the V_D-module structure on a V^0-module M is unique over $\mathbb{C}$.*

Theorem 2.11. *([Y1], Induced modules) Let W be an irreducible V^0-module. Then there exists a unique $\chi \in D^*$ such that W is contained in an irreducible χ-twisted V_D-module. If $D_W = 0$, then it is given by the induced module*

$$\mathrm{Ind}_{V^0}^{V_D} W := \bigoplus_{\alpha\in D} V^\alpha \boxtimes_{V^0} W.$$

Moreover, any irreducible χ-twisted V_D-module containing W as a V^0-submodule is isomorphic to $\mathrm{Ind}_{V^0}^{V_D} W$ above.

For a later purpose, we give a detailed description of the theorem above in the case of simple current super-extensions.

Theorem 2.12. *([Y2]) Let $V = V^0 \oplus V^1$ be a simple current super-extension of a simple rational C_2-cofinite VOA V^0 of CFT-type. For an irreducible V^0-module W,*
(i) If $V^1 \boxtimes_{V^0} W \not\simeq W$, then W is uniquely lifted to be an irreducible untwisted or $\mathbb{Z}_2$-twisted V-module $W \oplus (V^1 \boxtimes_{V^0} W)$.
(ii) If $V^1 \boxtimes_{V^0} W \simeq W$, then there exist exactly two inequivalent irreducible $\mathbb{Z}_2$-twisted V-module structure on W. If we write one of them by W^+, then the other one is given as the $\mathbb{Z}_2$-conjugate V-module of W^+.

Theorem 2.13. *([SY] [Y1], Lifting property of intertwining operators) Let M^i, $i = 1, 2, 3$, be irreducible D-stable V_D-modules and let W^i be irreducible V^0-submodules of M^i for $i = 1, 2, 3$. Then for a V^0-intertwining operator $I(\cdot, z)$ of type $W^1 \times W^2 \to W^3$ there is a V_D-intertwining operator $\tilde{I}(\cdot, z)$ of type $M^1 \times M^2 \to M^3$ such that $\tilde{I}(\cdot, z)|_{W^1 \otimes W^2} = I(\cdot, z)$. Therefore, there is a natural linear isomorphism*

$$\binom{M^3}{M^1\ M^2}_{V_D} \simeq \bigoplus_{\alpha \in D} \binom{V^\alpha \boxtimes_{V^0} W^3}{W^1 \qquad W^2}_{V^0},$$

where $\binom{M^3}{M^1\ M^2}_{V_D}$ denotes the space of V_D-intertwining operators of type $M^1 \times M^2 \to M^3$.

Corollary 2.14. *Let E be a subgroup of D and fix a coset decomposition $D = \sqcup_{i=1}^r (\alpha_i + E)$ of D. Then $V_E := \oplus_{\alpha \in E} V^\alpha$ is an E-graded simple current extension of V^0 and $V_{E+\alpha_i} := \oplus_{\beta \in E} V^{\alpha_i + \beta}$ are simple current V_E-modules. Therefore, we can view $V_D = \oplus_{i=1}^r V_{E+\alpha_i}$ as a D/E-graded simple current extension of V_E.*

Let us consider automorphisms on V_D. Let $\sigma \in \mathrm{Aut}(V^0)$ and $(X, Y_X(\cdot, z))$ a V^0-module. We can define the *σ-conjugate module* X^σ as follows. As a vector space, we set $X^\sigma = X$ and the vertex operator map on X^σ is defined by $Y_{X^\sigma}(a, z) := Y_X(\sigma a, z)$ for $a \in V^0$. It is clear that X^σ is irreducible if and only if X is an irreducible V^0-module. An irreducible V^0-module X is called *σ-stable* if its σ-conjugate X^σ is isomorphic to X as a V^0-module.

Remark 2.15. The definition of D-stable V_D-modules comes from the fact that if M is an irreducible D-stable V_D-module then its conjugate M^χ is isomorphic to M as a V_D-module for any $\chi \in D^*$.

We can construct the *σ-conjugate* $(V_D)^\sigma$ of V_D as follows. Let $Y_{V_D}(\cdot, z)$ be the vertex operator map on V_D. By definition, there are canonical linear isomorphisms $\psi_\alpha : V^\alpha \to (V^\alpha)^\sigma$, $\alpha \in D$, such that

$$Y_{(V^\alpha)^\sigma}(x^0, z)\psi_\alpha = \psi_\alpha Y_{V^\alpha}(\sigma x^0, z)$$

for all $x^0 \in V^0$. Then we define the vertex operator map $Y_{V_D}^\sigma(\cdot, z)$ on $(V_D)^\sigma = \oplus_{\alpha \in D} (V^\alpha)^\sigma$ by

$$Y_{V_D}^\sigma(\psi_\alpha x^\alpha, z)\psi_\alpha x^\beta := \psi_{\alpha+\beta} Y_{V_D}(x^\alpha, z) x^\beta$$

for $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$. Then one can easily verify that $((V_D)^\sigma, Y_{V_D}^\sigma(\cdot, z))$ also forms a D-graded simple current extension of V^0. The following lifting property is established in [Sh] by using the uniqueness of VOA structure on V_D.

Theorem 2.16. *([Sh], Lifting property of automorphisms) Let $\sigma \in \mathrm{Aut}(V^0)$ such that $(V_D)^\sigma \simeq V_D$ as a V^0-module. Then there is a lifting $\tilde{\sigma} \in \mathrm{Aut}(V_D)$ such that $\tilde{\sigma} V^0 = V^0$ and $\tilde{\sigma}|_{V^0} = \sigma$. The lifting $\tilde{\sigma}$ is unique up to multiples of elements in $D^* \subset \mathrm{Aut}(V_D)$. This assertion still holds if $D = \mathbb{Z}_2$ and $V_D = V^0 \oplus V^1$ is a simple current super-extension of V^0.*

As we have seen above, simple current extensions have many good properties. At last of this section, we present the following extension property of simple current extensions.

Theorem 2.17. *([Y2], Extension property of simple current extensions) Let $V^{(0,0)}$ be a simple rational C_2-cofinite VOA of CFT-type, and let D_1, D_2 be finite abelian groups. Assume that we have a set of inequivalent irreducible simple current $V^{(0,0)}$-modules $\{V^{(\alpha,\beta)} \mid (\alpha,\beta) \in D_1 \oplus D_2\}$ with $D_1 \oplus D_2$-graded fusion rules $V^{(\alpha_1,\beta_1)} \boxtimes_{V^{(0,0)}} V^{(\alpha_2,\beta_2)} = V^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}$ for any $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in D_1 \oplus D_2$.*
(i) Further assume that all $V^{(\alpha,\beta)}$, $(\alpha,\beta) \in D_1 \oplus D_2$, have integral top weights and we have D_1- and D_2-graded simple current extensions $V_{D_1} = \oplus_{\alpha\in D_1} V^{(\alpha,0)}$ and $V_{D_2} = \oplus_{\beta \in D_2} V^{(0,\beta)}$. Then $V_{D_1\oplus D_2} := \oplus_{(\alpha,\beta)\in D_1\oplus D_2} V^{(\alpha,\beta)}$ possesses a unique structure of a simple vertex operator algebra as a $D_1 \oplus D_2$-graded simple current extension of $V^{(0,0)}$.
(ii) In the case of $D_2 = \mathbb{Z}_2 = \{0,1\}$, further assume that all $V^{(\alpha,0)}$, $\alpha \in D_1$, have integral top weight, all $V^{(\alpha,1)}$, $\alpha \in D_1$, have half-integral top weight, $V_{D_1} = \oplus_{\alpha\in D_1} V^{(\alpha,0)}$ is a D_1-graded simple current extension of $V^{(0,0)}$, and $V_{D_2} = V^{(0,0)} \oplus V^{(0,1)}$ is a simple current super-extension of $V^{(0,0)}$. Then $V_{D_1\oplus D_2} = \oplus_{(\alpha,\beta)\in D_1\oplus D_2} V^{(\alpha,\beta)}$ has a unique structure of a simple vertex operator superalgebra with even part $\oplus_{\alpha\in D_1} V^{(\alpha,0)}$ and odd part $\oplus_{\beta\in D_1} V^{(\beta,1)}$ as a simple current super-extension of V_{D_1}.

3 Miyamoto involution and its centralizer

Let us denote by $L(c,h)$ the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h. It is shown in [FZ] that $L(c,0)$ has a structure of a simple VOA. Here we consider the first unitary Virasoro VOA $L(1/2,0)$. It is proved in [DMZ] [W] that $L(1/2,0)$ is a rational C_2-cofinite VOA of CFT-type and has exactly three irreducible modules, $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$. Their fusion rules have also been computed and are as follows:

$$\begin{aligned} &L(1/2,1/2)\times L(1/2,1/2) = L(1/2,0),\\ &L(1/2,1/2)\times L(1/2,1/16) = L(1/2,1/16),\\ &L(1/2,1/16)\times L(1/2,1/16) = L(1/2,0)+L(1/2,1/2). \end{aligned} \tag{3.1}$$

First, we present an explicit realization of $L(1/2,0)$ and its modules.

3.1 Ising model

In this section we give an explicit construction of the Ising model SVOA $L(1/2,0)\oplus L(1/2,1/2)$ and its $\mathbb{Z}_2$-twisted modules $L(1/2,1/16)^{\pm}$. This construction is well-known and the most of contents in this section can be found in [FFR], [FRW] and [KR].

Let $\mathcal{A}_\psi$ be the algebra generated by $\{\psi_k \mid k\in\mathbb{Z}+\frac{1}{2}\}$ subject to the defining relation

$$[\psi_m,\psi_n]_+ := \psi_m\psi_n+\psi_n\psi_m = \delta_{m+n,0},\quad m,n\in\mathbb{Z}+\tfrac{1}{2},$$

and denote a subalgebra of $\mathcal{A}_\psi$ generated by $\{\psi_k \mid k\in\mathbb{Z}+\frac{1}{2}, k>0\}$ by $\mathcal{A}_\psi^+$. Let $\mathbb{C}\mathbb{1}$ be a trivial $\mathcal{A}_\psi^+$-module. Define a canonical induced $\mathcal{A}_\psi$-module M by

$$M := \mathrm{Ind}_{\mathcal{A}_\psi^+}^{\mathcal{A}_\psi}\mathbb{C}\mathbb{1} = \mathcal{A}_\psi\otimes_{\mathcal{A}_\psi^+}\mathbb{C}\mathbb{1}.$$

Consider the generating series

$$\psi(z) := \sum_{n\in\mathbb{Z}}\psi_{n+\frac{1}{2}}z^{-n-1}.$$

Since $[\psi(z),\psi(w)]_+ = z^{-1}\delta(\frac{w}{z})$, $\psi(z)$ is local with itself. So we can consider a subalgebra of a local system on M generated by $\psi(z)$ and $I(z)=\mathrm{id}_M$. By a direct calculation, one sees that the component operators of the generating series

$$\frac{1}{2}\psi(z)\circ_{-2}\psi(z) := \frac{1}{2}\mathrm{Res}_{z_0}\{(z_0-z)^{-2}\psi(z_0)\psi(z)+(-z+z_0)^{-2}\psi(z_0)\psi(z)\}$$

defines a representation of the Virasoro algebra on M with central charge $1/2$, where $\circ_n$ denotes the n-th normal ordered product defined in [Li1]. It follows from [KR] that M as a Vir-module is isomorphic to $L(1/2,0)\oplus L(1/2,1/2)$. Therefore, there is a unique simple vertex operator superalgebra structure on M such that $Y_M(\psi_{-\frac{1}{2}}\mathbb{1},z)=\psi(z)$.

Theorem 3.1. *On M, there is a unique simple vertex operator superalgebra structure $(M,Y_M(\cdot,z),\mathbb{1},\frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\mathbb{1})$ such that $Y_M(\psi_{-\frac{1}{2}}\mathbb{1},z)=\psi(z)$.*

Another unitary Vir-module $L(1/2,1/16)$ is realized as follows. Let $\mathcal{A}_\phi$ be the algebra generated by $\{\phi_n \mid n\in\mathbb{Z}\}$ with defining relation

$$[\phi_m,\phi_n]_+ = \delta_{m+n,0},\quad m,n\in\mathbb{Z}.$$

Let $\mathcal{A}_\phi^+$ be a subalgebra of $\mathcal{A}_\phi$ generated by $\{\phi_n | n > 0\}$ and let $\mathbb{C}v_0$ be a trivial one-dimensional $\mathcal{A}_\phi^+$-module. Then set $N = \mathrm{Ind}_{\mathcal{A}_\phi^+}^{\mathcal{A}_\phi} \mathbb{C}v_0$ as we did previously. We can find an action of the Virasoro algebra on N. Consider the generating series

$$\phi(z) := \sum_{n\in\mathbb{Z}} \phi_n z^{-n-\frac{1}{2}}.$$

By direct calculations one can show that $\phi(z)$ is local with itself. Consider a local system on N containing $\phi(z)$. Since the powers of z in $\phi(z)$ lie in $\mathbb{Z} + \frac{1}{2}$, we have to use the $\mathbb{Z}_2$-twisted normal ordered product in [Li2]. Define a generating series $L(z)$ of operators on N by

$$\begin{aligned} L(z) &:= \phi(z) \circ_{-2} \phi(z) \\ &= \frac{1}{2}\mathrm{Res}_{z_0}\mathrm{Res}_{z_1} z_0^{-2} \left(\frac{z_1 - z_0}{z}\right)^{\frac{1}{2}} \\ &\qquad \times \left\{ z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)\phi(z_1)\phi(z) + z_0^{-1}\delta\left(\frac{-z + z_1}{z_0}\right)\phi(z)\phi(z_1)\right\}, \end{aligned}$$

where $\circ_n$ above denotes the n-th normal ordered product in a $\mathbb{Z}_2$-twisted local system on N (cf. [Li2]). Then by a direct computation we find that the component operators of $L(z)$ defines a representation of the Virasoro algebra on N with central charge $1/2$. Set $v_{1/16}^{\pm} := \phi_0 \mathbb{1} \pm (1/\sqrt{2})\mathbb{1}$. Then $v_{1/16}^{\pm}$ are highest weight vectors for the Virasoro algebra and the following decomposition is shown in [KR]:

$$N = L(1/2, 1/16)^+ \oplus L(1/2, 1/16)^-,$$

where $L(1/2, 1/16)^{\pm}$ are highest weight Vir-module generated by $v_{1/16}^{\pm}$, respectively.

Theorem 3.2. *The following $\mathbb{Z}_2$-twisted Jacobi identity holds on N:*

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_N(a, z_1)Y_N(b, z_2) - (-1)^{\varepsilon(a,b)} z_0^{-1}\left(\frac{-z_2 + z_1}{z_0}\right) Y_N(b, z_2)Y_N(a, z_1) \\ &= z_1^{-1}\left(\frac{z_2 + z_0}{z_1}\right)\left(\frac{z_2 + z_0}{z_1}\right)^{\varepsilon(a,a)/2} Y_N(Y_M(a, z_0)b, z_2), \end{aligned}$$

where $a, b \in M = L(1/2, 0) \oplus L(1/2, 1/2)$ and $\varepsilon(\cdot,\cdot)$ denotes the standard parity function. Therefore, the vertex operator map $Y_N(\cdot, z)$ defines inequivalent irreducible $\mathbb{Z}_2$-twisted $L(1/2, 0) \oplus L(1/2, 1/2)$-module structures on $L(1/2, 1/16)^{\pm}$.

3.2 Miyamoto involution

Let $(V, Y_V(\cdot, z), \mathbb{1}, \omega)$ be a VOA. A vector $e \in V$ is called a *conformal vector* if its component operators $Y_V(e, z) = \sum_{n\in\mathbb{Z}} e_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L^e(n) z^{-n-2}$ generate a representation

of the Virasoro algebra on V:

$$[L^e(m), L^e(n)] = (m-n)L^e(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}c_e.$$

The scalar c_e is called *central charge* of a conformal vector e. We denote by $\mathrm{Vir}(e)$ the sub VOA generated by e. If $\mathrm{Vir}(e)$ is a rational VOA, then e is called a *rational conformal vector*. A decomposition $\omega = e + (\omega - e)$ is called *orthogonal* if both e and $\omega - e$ are conformal vectors and their component operators are mutually commutative.

Now assume that $e \in V$ is a rational conformal vector with central charge 1/2. Then $\mathrm{Vir}(e)$ is isomorphic to $L(1/2, 0)$. Since $L(1/2, 0)$ is rational, we can decompose V into a direct sum of irreducible $\mathrm{Vir}(e)$-modules as follows:

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16),$$

where $V_e(h)$, $h \in \{0, 1/2, 1/16\}$, denotes the sum of all irreducible $\mathrm{Vir}(e)$-submodules of V isomorphic to $L(1/2, h)$. By the fusion rules (3.1), we have the following grading structure (cf. [M1]):

$$\begin{aligned} &V_e(0) \cdot V_e(h) \subset V_e(h), \quad h = 0, 1/2, 1/16, \quad V_e(1/2) \cdot V_e(1/2) \subset V_e(0), \\ &V_e(1/2) \cdot V_e(1/16) \subset V_e(1/16), \quad V_e(1/16) \cdot V_e(1/16) \subset V_e(0) \oplus V_e(1/2). \end{aligned}$$

Therefore, if $V_e(1/16) \neq 0$, then the linear map

$$\tau_e := \ 1 \quad \text{on} \quad V_e(0) \oplus V_e(1/2), \quad -1 \quad \text{on} \quad V_e(1/16)$$

defines an involutive automorphism on V (cf. [M1]). We call τ_e the *first Miyamoto involution* or simply *Miyamoto involution* associated to a conformal vector e. If $V_e(1/16) = 0$, then we can also define another involution as follows (cf. [M1]):

$$\sigma_e := \ 1 \quad \text{on} \quad V_e(0), \quad -1 \quad \text{on} \quad V_e(1/2).$$

We call σ_e the *second Miyamoto involution* associated to e.

Remark 3.3. It is shown in [C] and [M1] that the Miyamoto involution τ_e associated to a conformal vector e of the moonshine VOA [FLM] with central charge 1/2 defines a 2A-involution of the Monster.

3.3 Commutant superalgebra

We keep the same notation as in the previous subsection. Let V be a simple VOA of CFT-type and $e \in V$ a rational conformal vector with central charge 1/2. Set $T_e(h) := \{v \in V \mid L^e(0)v = h \cdot v\}$ for $h = 0, 1/2, 1/16$. Then $T_e(h)$ is a space of highest weight vectors

for $\mathrm{Vir}(e)$ and is canonically isomorphic to $\mathrm{Hom}_{\mathrm{Vir}(e)}(L(1/2,h),V)$ for $h=0,1/2,1/16$. Therefore, we have a decomposition as follows:

$$V=L(1/2,0)\otimes T_e(0)\oplus L(1/2,1/2)\otimes T_e(1/2)\oplus L(1/2,1/16)\otimes T_e(1/16).$$

Lemma 3.4. *A decomposition $\omega=e+(\omega-e)$ is orthogonal.*

Proof: We compute $e_{(1)}\omega_{(2)}e$.

$$\begin{aligned} e_{(1)}\omega_{(2)}e &= \omega_{(2)}e_{(1)}e+[e_{(1)},\omega_{(2)}]e \\ &= 2\omega_{(2)}e-[\omega_{(2)},e_{(1)}]e \\ &= 2\omega_{(2)}e-\{(\omega_{(0)}e)_{(3)}+2(\omega_{(1)}e)_{(2)}+(\omega_{(2)}e)_{(1)}\}e \\ &= 2\omega_{(2)}e-(\omega_{(2)}e)_{(1)}e. \end{aligned}$$

By the skew-symmetry, we have $(\omega_{(2)}e)_{(1)}e=e_{(1)}\omega_{(2)}e-\omega_{(0)}e_{(2)}\omega_{(2)}e$. Since $e_{(2)}\omega_{(2)}e\in V_0=\mathbb{C}\mathbb{1}$, $\omega_{(0)}e_{(2)}\omega_{(2)}e=0$ and so $(\omega_{(2)}e)_{(1)}e=e_{(1)}\omega_{(2)}e$. Substituting this into the equality above, we get $e_{(1)}\omega_{(2)}e=\omega_{(2)}e$. Namely, $\omega_{(2)}e$ is an eigenvector for $e_{(1)}$ with eigenvalue 1. Since V is a module for $\mathrm{Vir}(e)$, there is no eigenvector with $e_{(1)}$-weight 1. Hence $\omega_{(2)}e=0$. Then the assertion follows from Theorem 5.1 of [FZ]. ∎

Recall the commutant subalgebra $\mathrm{Com}_V(\mathrm{Vir}(e)):=\ker_V e_{(0)}$ defined in [FZ]. By the lemma above, $(T_e(0),\omega-e)$ forms a sub VOA of V whose action on V is commutative with that of $\mathrm{Vir}(e)$ on V. In particular, $T_e(h)$, $h=0,1/2,1/16$, are $T_e(0)$-modules.

Proposition 3.5. *(1) $T_e(0)=\ker_V e_{(0)}=\mathrm{Com}_V(\mathrm{Vir}(e))$ is a simple sub VOA with the Virasoro vector $\omega-e$.*
(2) $T_e(1/2)$ is an irreducible $T_e(0)$-module.
(3) $\mathrm{Vir}(e)=\ker_V(\omega-e)_{(0)}=\mathrm{Com}_V(T_e(0))$.

Proof: (1): Let $v\in V$. Since $e_{(1)}v=0$ implies $e_{(0)}v=0$, $T_e(0)=\ker_V e_{(1)}=\ker_V e_{(0)}$. So we only need to show that $T_e(0)$ is simple. Since V is simple, the τ_e-orbifold $V^{\langle\tau_e\rangle}=V_e(0)\oplus V_e(1/2)$ is simple. Then the σ_e-orbifold $(V^{\langle\tau_e\rangle})^{\langle\sigma_e\rangle}=V_e(0)$ is also simple. Since $\mathrm{Vir}(e)\otimes T_e(0)\ni a\otimes b\mapsto a_{(-1)}b\in V_e(0)$ is an isomorphism of VOAs, $T_e(0)$ is also simple.
(2): Since both $V^{\langle\tau_e\rangle}=V_e(0)\oplus V_e(1/2)$ and $V_e(0)$ are simple VOAs, $V_e(1/2)$ is an irreducible $V_e(0)$-module. So $T_e(1/2)$ is also irreducible.
(3): As $\omega-e$ is a conformal vector, $\ker_V(\omega-e)_{(0)}$ is generally contained in $\ker_V(\omega-e)_{(1)}$. On the other hand, since V is of CFT-type, $\ker_V(\omega-e)_{(1)}=\mathrm{Vir}(e)$. Then

$$\mathrm{Vir}(e)\subset\mathrm{Com}_V(\mathrm{Com}_V(\mathrm{Vir}(e))=\ker_V(\omega-e)_{(0)}$$

implies $\mathrm{Vir}(e)=\ker_V(\omega-e)_{(0)}$. ∎

Theorem 3.6. *Suppose that $T_e(1/2) \neq 0$. Then there exists a simple SVOA structure on $T_e(0) \oplus T_e(1/2)$ such that the even part of a tensor product of SVOAs*

$$\{L(1/2,0) \oplus L(1/2,1/2)\} \otimes \{T_e(0) \oplus T_e(1/2)\}$$

is isomorphic to $V_e(0) \oplus V_e(1/2)$ as a VOA.

Proof: We shall define vertex operators on an abstract space $T_e(0) \oplus T_e(1/2)$. First, we show an existence of a $T_e(0)$-intertwining operator of type $T_e(1/2) \times T_e(1/2) \to T_e(0)$. Write $Y_V(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$ and $Y_V(e, z) = \sum_{n\in\mathbb{Z}} L^e(n) z^{-n-2}$. Since $L(0) - L^e(0)$ semisimply acts on both $T_e(0)$ and $T_e(1/2)$, we can take bases $\{a^\gamma \mid \gamma \in \Gamma\}$ and $\{u^\lambda \mid \lambda \in \Lambda\}$ of $T_e(0)$ and $T_e(1/2)$, respectively, consisting of eigenvectors for $L(0) - L^e(0)$. Let $\pi_\gamma : V_e(0) \to L(1/2,0) \otimes a^\gamma$, $\gamma \in \Gamma$, be a projection map. For $\gamma \in \Gamma$ and $\lambda, \mu \in \Lambda$, we define a linear operator $I^\gamma_{\lambda\mu}(\cdot, z)$ of type $L(1/2,1/2) \times L(1/2,1/2) \to L(1/2,0) \otimes a^\gamma$ by

$$\begin{aligned} I^\gamma_{\lambda\mu}(x,z)y \ &:= z^{-L(0)+L^e(0)} \pi_\gamma Y(z^{L(0)-L^e(0)} x \otimes u^\lambda, z) z^{L(0)-L^e(0)} y \otimes u^\mu \\ &= z^{-|\gamma|+|\lambda|+|\mu|} \pi_\gamma Y(x \otimes u^\lambda, z) y \otimes u^\mu, \end{aligned}$$

for $x, y \in L(1/2,1/2)$, where $|\lambda|$, $|\mu|$ and $|\gamma|$ denote the $(L(0) - L^e(0))$-weight of u^λ, u^μ and a^γ, respectively. Then by [DL] [M1] the operator $I^\gamma_{\lambda\mu}(\cdot, z)$ is an $L(1/2,0)$-intertwining operator of type $L(1/2,1/2) \times L(1/2,1/2) \to L(1/2,0)$. Since the space of intertwining operators of that type is one-dimensional, each $I^\gamma_{\lambda\mu}(\cdot, z)$ is proportional to the vertex operator map $Y_M(\cdot, z)$ on the SVOA $M = L(1/2,0) \oplus L(1/2,1/2)$ which we constructed explicitly in Section 3.1. Thus there exist scalars $c^\gamma_{\lambda\mu} \in \mathbb{C}$ such that $I^\gamma_{\lambda\mu}(\cdot, z) = c^\gamma_{\lambda\mu} Y_M(\cdot, z)$. Then the vertex operator of $x \otimes u^\lambda \in L(1/2,1/2) \otimes T_e(1/2)$ on $V_e(1/2)$ can be written as follows:

$$Y_V(x \otimes u^\lambda, z) y \otimes u^\mu = Y_M(x, z) y \otimes \sum_{\gamma \in \Gamma} c^\gamma_{\lambda\mu} a^\gamma z^{|\gamma|-|\lambda|-|\mu|}.$$

Thus, by setting $J(u^\lambda, z) u^\mu := \sum_{\gamma\in\Gamma} c^\gamma_{\lambda\mu} a^\gamma z^{|\gamma|-|\lambda|-|\mu|}$, we obtain a decomposition

$$Y_V(x \otimes u^\lambda, z) y \otimes u^\mu = Y_M(x, z) y \otimes J(u^\lambda, z) u^\mu$$

for $x \otimes u^\lambda$, $y \otimes u^\mu \in L(1/2,1/2) \otimes V_e(1/2)$. We claim that $J(\cdot, z)$ is a $T_e(0)$-intertwining operator of type $T_e(1/2) \times T_e(1/2) \to T_e(0)$. It is obvious that $J(u, z)v$ contains finitely many negative powers of z and the $(\omega - e)_{(0)}$-derivation property $J((\omega - e)_0 u, z)v = \frac{d}{dz} J(u, z)v$ hold for all $u, v \in T_e(1/2)$. So we should show that $J(\cdot, z)$ satisfies both the commutativity and the associativity. Let $a \in T_e(0)$ and $u, v \in T_e(1/2)$ be arbitrary elements. Then the commutativity of vertex operators on V gives

$$\begin{aligned} &(z_1 - z_2)^N Y_V(\mathbb{1} \otimes a, z_1) Y_V(\psi_{-\frac{1}{2}} \mathbb{1} \otimes u, z_2) \psi_{-\frac{1}{2}} \mathbb{1} \otimes v \\ &= (z_1 - z_2)^N Y_V(\psi_{-\frac{1}{2}} \mathbb{1} \otimes u, z_2) Y_V(\mathbb{1} \otimes a, z_1) \psi_{-\frac{1}{2}} \mathbb{1} \otimes v \end{aligned}$$

for sufficiently large N. Rewriting the equality above we get

$$\begin{aligned}&(z_1-z_2)^N Y_M(\psi_{-\frac{1}{2}}\mathbb{1},z_2)\psi_{-\frac{1}{2}}\mathbb{1}\otimes Y_{T_e(0)}(a,z_1)J(u,z_2)v\\&=(z_1-z_2)^N Y_M(\psi_{-\frac{1}{2}}\mathbb{1},z_2)\psi_{-\frac{1}{2}}\mathbb{1}\otimes J(u,z_2)Y_{T_e(1/2)}(a,z_1)v,\end{aligned}$$

where $Y_{T_e(0)}(a,z)$ and $Y_{T_e(1/2)}(\cdot,z)$ denote the vertex operator of $a\in T_e(0)$ on $T_e(0)$ and $T_e(1/2)$, respectively. By comparing the coefficients of $(\psi_{-\frac{1}{2}}\mathbb{1})_{(0)}\psi_{-\frac{1}{2}}\mathbb{1}=\mathbb{1}$, we get the commutativity:

$$(z_1-z_2)^N Y_{T_e(0)}(a,z_1)J(u,z_2)v=(z_1-z_2)^N J(u,z_2)Y_{T_e(1/2)}(a,z_1)v.$$

Similarly, by considering coefficients of $Y_V(Y_V(\mathbb{1}\otimes a,z_0)\psi_{-\frac{1}{2}}\mathbb{1}\otimes u,z_2)\psi_{-\frac{1}{2}}\mathbb{1}\otimes v$ in V, we obtain the associativity:

$$(z_0+z_2)^N Y_{T_e(0)}(a,z_0+z_2)J(u,z_2)v=(z_2+z_0)^N J(Y_{T_e(1/2)}(a,z_0)u,z_2)v.$$

Hence, $J(\cdot,z)$ is a $T_e(0)$-intertwining operator of the desired type.

Using $Y_V(\cdot,z)$ and $J(\cdot,z)$, we introduce a vertex operator map $\hat{Y}(\cdot,z)$ on $T_e(0)\oplus T_e(1/2)$. Let $a,b\in T_e(0)$ and $u,v\in T_e(1/2)$. We define

$$\begin{aligned}&\mathbb{1}\otimes\hat{Y}(a,z)b:=Y_V(\mathbb{1}\otimes a,z)\mathbb{1}\otimes b,\quad \psi_{-\frac{1}{2}}\mathbb{1}\otimes\hat{Y}(a,z)u:=Y_V(\mathbb{1}\otimes a,z)\psi_{-\frac{1}{2}}\mathbb{1}\otimes u,\\&\psi_{-\frac{1}{2}}\mathbb{1}\otimes\hat{Y}(u,z)a:=e^{z(L(-1)-L^e(-1))}Y_V(\mathbb{1}\otimes a,z)\psi_{-\frac{1}{2}}\mathbb{1}\otimes u,\quad \hat{Y}(u,z)v:=J(u,z)v.\end{aligned}$$

Then all $\hat{Y}(\cdot,z)$ are $T_e(0)$-intertwining operators. We note that $\hat{Y}(\cdot,z)$ satisfies the vacuum condition:

$$\hat{Y}(x,z)\mathbb{1}\in x+(T_e(0)\oplus T_e(1/2))\,[[z]]z$$

for any $x\in T_e(0)\oplus T_e(1/2)$. Hence, to prove that $T_e(0)\oplus T_e(1/2)$ is a simple SVOA, it is sufficient to show that the vertex operator map $\hat{Y}(\cdot,z)$ defined above satisfies the commutativity. By our definition, the vertex operator map $Y_V(a\otimes x,z)$ of $a\otimes x\in L(1/2,h)\otimes T_e(h)=V_e(h)$, $h=0,1/2$, can be written as $Y_M(a,z)\otimes\hat{Y}(x,z)$. Because of our manifest construction of $Y_M(\cdot,z)$ in Section 3.1, we can perform explicit computations of the vertex operator $Y_M(\cdot,z)$ on $L(1/2,0)\oplus L(1/2,1/2)$. Therefore, by comparing the coefficients of vertex operators on V, we can prove that $\hat{Y}(\cdot,z)$ satisfies the (super-)commutativity. Thus, by our definition, $(T_e(0)\oplus T_e(1/2),\hat{Y}(\cdot,z),\mathbb{1},\omega-e)$ carries a structure of a simple SVOA. The rest of the assertion is now clear. ∎

Remark 3.7. There is another proof of Theorem 3.6 in [Hö1]. In [Hö1], he assumed the existence of a positive definite invariant bilinear form on a real form of V. However, our argument does not need the assumption on the unitary form.

Since $\tau_e^2 = 1$ on V, the space $V_e(1/16)$ is an irreducible $V^{\langle \tau_e \rangle}$-module. As a $(V^{\langle \tau_e \rangle})^{\langle \sigma_e \rangle} = \mathrm{Vir}(e) \otimes T_e(0)$-module, $V_e(1/16)$ can be written as $L(1/2, 1/16) \otimes T_e(1/16)$. It is not clear that $T_e(1/16)$ is irreducible under $T_e(0)$. However, we can prove that it is irreducible under $T_e(0) \oplus T_e(1/2)$.

Theorem 3.8. *Suppose that $V_e(1/2) \neq 0$ and $V_e(1/16) \neq 0$. Then $T_e(1/16)$ carries a structure of an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. Moreover, $V_e(1/16)$ is isomorphic to a tensor product of an irreducible $\mathbb{Z}_2$-twisted $L(1/2, 0) \oplus L(1/2, 1/2)$-module $L(1/2, 1/16)^+$ and an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module $T_e(1/16)$.*

Proof: The idea of the proof is the same as that of Theorem 3.6. Computing vertex operators on $L(1/2, 1/16)^+$ and then comparing the coefficients in V, we will reach the assertion. Denote by $Y_N(\cdot, z)$ the vertex operator map on the $\mathbb{Z}_2$-twisted $L(1/2, 0) \oplus L(1/2, 1/2)$-module $L(1/2, 1/16)^+$ as we constructed in Section 3.1. Let $a \otimes b \in L(1/2, h) \otimes T_e(h)$ with $h = 0$ or $1/2$ and $x \otimes y \in L(1/2, 1/16) \otimes T_e(1/16)$. As we did before, we can find $T_e(0)$-intertwining operators $Y_{T_e(h) \times T_e(1/16)}(\cdot, z)$ of types $T_e(h) \times T_e(1/16) \to T_e(1/16)$ such that

$$Y_V(a \otimes b, z) x \otimes y = Y_N(a, z) x \otimes Y_{T_e(h) \times T_e(\frac{1}{16})}(b, z) y. \tag{3.2}$$

Define $\hat{Y}(b, z) y := Y_{T_e(h) \times T_e(\frac{1}{16})}(b, z) y$ for $b \in T_e(h)$, $h = 0, 1/2$ and $y \in T_e(1/16)$. By direct computations, we can prove that the $\mathbb{Z}_2$-twisted Jacobi identity for $Y_N(\cdot, z)$ together with the Jacobi identity for $Y_V(\cdot, z)$ gives the $\mathbb{Z}_2$-twisted Jacobi identity for $\hat{Y}(\cdot, z)$. Thus, $(T_e(1/16), \hat{Y}(\cdot, z))$ is a $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. Since $V_e(1/16) = L(1/2, 1/16) \otimes T_e(1/16)$ is irreducible under $V_e(0) \oplus V_e(1/2)$, the irreducibility of $T_e(1/16)$ is obvious. ∎

3.4 One point stabilizer

In the rest of this section we will work the following setup:

Hypothesis 1.
(1) V is a holomorphic VOA of CFT-type.
(2) e is a rational conformal vector of V with central charge $1/2$.
(3) $V_e(h) \neq 0$ for $h = 0, 1/2, 1/16$.
(4) $V_e(0)$ and $T_e(0)$ are rational C_2-cofinite VOAs of CFT-type.
(5) $V_e(1/16)$ is a simple current $V^{\langle \tau_e \rangle} = V_e(0) \oplus V_e(1/2)$-module.
(6) $T_e(1/2)$ is a simple current $T_e(0)$-module.

Define the one-point stabilizer by $C_{\mathrm{Aut}(V)}(e) := \{\rho \in \mathrm{Aut}(V) \mid \rho(e) = e\}$. Then by $\tau_{\rho(e)} = \rho \tau_e \rho^{-1}$ for any $\rho \in \mathrm{Aut}(V)$, we have $C_{\mathrm{Aut}(V)}(e) \subset C_{\mathrm{Aut}(V)}(\tau_e)$, where $C_{\mathrm{Aut}(V)}(\tau_e)$ denotes the centralizer of an involution $\tau_e \in \mathrm{Aut}(V)$.

Lemma 3.9. *There are group homomorphisms $\psi_1 : C_{\mathrm{Aut}(V)}(e) \to C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ and $\psi_2 : C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \to \mathrm{Aut}(T_e(0))$ such that $\ker(\psi_1) = \langle\tau_e\rangle$ and $\ker(\psi_2) = \langle\sigma_e\rangle$.*

Proof: Let $\rho \in C_{\mathrm{Aut}(V)}(e)$. Then ρ preserves the space of highest weight vectors $T_e(h)$ where $h \in \{0, 1/2, 1/16\}$. Then we can define the actions of ρ on the space of highest weight vectors $T_e(h)$ and the components $V_e(h)$ for $h \in \{0, 1/2, 1/16\}$. In particular, we have group homomorphisms $\psi_1 : C_{\mathrm{Aut}(V)}(e) \to C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$ and $\psi_2 : C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \to \mathrm{Aut}(T_e(0))$ by a natural way. Assume that $\psi_1(\rho) = \mathrm{id}_{V^{\langle\tau_e\rangle}}$ for $\rho \in C_{\mathrm{Aut}(V)}(e)$. Since $\rho \in C_{\mathrm{Aut}(V)}(\tau_e)$, ρ acts on $V_e(1/16)$ and commutes with the action of $V^{\langle\tau_e\rangle} = V_e(0) \oplus V_e(1/2)$ on its module $V_e(1/16)$. Therefore, ρ on $V_e(1/16)$ is a scalar by Schur's lemma and hence $\rho \in \langle\tau_e\rangle \subset C_{\mathrm{Aut}(V)}(\tau_e)$. Similarly, if $\psi_1(\rho') = \mathrm{id}_{T_e(0)}$ for $\rho' \in C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$, then $\rho' \in \langle\sigma_e\rangle \subset C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. ∎

Theorem 3.10. *Under Hypothesis 1, $V^{\langle\tau_e\rangle}$ has exactly four inequivalent irreducible modules, $V^{\langle\tau_e\rangle}$, $V_e(1/16)$, $W^0 := L(1/2, 0) \otimes T_e(1/2) \oplus L(1/2, 1/2) \otimes T_e(0)$ and*

$$W^1 := V_e(1/16) \boxtimes_{V^{\langle\tau_e\rangle}} W^0.$$

Proof: Note that $V_e(0) = \mathrm{Vir}(e) \otimes T_e(0)$ and $V^{\langle\tau_e\rangle}$ are simple rational C_2-cofinite VOAs of CFT-type under Hypothesis 1. Therefore, we can apply a theory of fusion products here. Since $V = V^{\langle\tau_e\rangle} \oplus V_e(1/16)$ is a $\mathbb{Z}_2$-graded simple current extension of $V^{\langle\tau_e\rangle}$, every irreducible $V^{\langle\tau_e\rangle}$-module is lifted to be either an irreducible V-module or an irreducible τ_e-twisted V-module. Moreover, the τ_e-twisted V-module is unique up to isomorphism by Theorem 10.3 of [DLM2]. Consider a $V_e(0)$-module $L(1/2, 1/2) \otimes T_e(0)$. Since $T_e(1/2)$ is a simple current $T_e(0)$-module, the space

$$W^0 = L(1/2, 1/2) \otimes T_e(0) \oplus L(1/2, 0) \otimes T_e(1/2)$$

has a unique structure of an irreducible $V^{\langle\tau_e\rangle}$-module by Theorem 2.11. We note that the top weight of W^0 is half-integral. Thus the induced module

$$W = W^0 \oplus W^1, \quad W^1 = V_e(1/16) \boxtimes_{V^{\langle\tau_e\rangle}} W^0,$$

becomes an irreducible τ_e-twisted V-module again by Theorem 2.11. Therefore, $V^{\langle\tau_e\rangle}$ has exactly four irreducible modules as in the assertion. Finally we remark that $V^{\langle\tau_e\rangle}$, $V_e(1/16)$ and W^1 have integral top weights. ∎

By the fusion rules (3.1), we note that W^1 as a $\mathrm{Vir}(e)$-module is a direct sum of copies of $L(1/2, 1/16)$. Set the space of highest weight vectors of W^1 by $Q_e(1/16) := \{v \in W^1 \mid L^e(0)v = (1/16) \cdot v\}$. Then as a $\mathrm{Vir}(e) \otimes T_e(0)$-module, $W^1 \simeq L(1/2, 1/16) \otimes Q_e(1/16)$. In this case, we can also verify that the space $Q_e(1/16)$ naturally carries an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module structure.

Proposition 3.11. *If the $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module $T_e(1/16)$ is irreducible as a $T_e(0)$-module, then its $\mathbb{Z}_2$-conjugate is isomorphic to $Q_e(1/16)$ as a $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. In this case there are three irreducible $T_e(0)$-modules, $T_e(0)$, $T_e(1/2)$ and $T_e(1/16)$. Conversely, if $T_e(1/16)$ as a $T_e(0)$-module is not irreducible, then so is $Q_e(1/16)$ and in this case there are six inequivalent irreducible $T_e(0)$-modules.*

Proof: Assume that $T_e(1/16)$ is irreducible as a $T_e(0)$-module. Then its $\mathbb{Z}_2$-conjugate is not isomorphic to $T_e(1/16)$ as a $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. We denote the $\mathbb{Z}_2$-conjugate of $T_e(1/16)$ by $T_e(1/16)^-$. It is shown in Theorem 2.12 that every irreducible $T_e(0)$-module is lifted to be either an irreducible $T_e(0) \oplus T_e(1/2)$-module or an irreducible $\mathbb{Z}_2$-twisted $T_e(0) \oplus T_e(1/2)$-module. Then by the classification of irreducible $V^{\langle\tau_e\rangle}$-modules in Theorem 3.10, we see that any $\mathbb{Z}_2$-twisted irreducible $T_e(0) \oplus T_e(1/2)$-module is isomorphic to one and only one of $T_e(1/16)$ and $T_e(1/16)^- = Q_e(1/16)$.

Conversely, if $T_e(1/16)$ is not irreducible, then it is a direct sum of two inequivalent irreducible $T_e(0)$-module as $T_e(1/2)$ is a simple current $T_e(0)$-module. Then $Q_e(1/16)$ is also a direct sum of two inequivalent irreducible $T_e(0)$-modules and $Q_e(1/16) \not\simeq T_e(1/16)$ as $T_e(0)$-modules because of the classification of irreducible $V^{\langle\tau_e\rangle}$-modules. ∎

Corollary 3.12. *If $T_e(1/16)$ is irreducible as a $T_e(0)$-module, then $V^{\langle\tau_e\rangle} \oplus W^1$ is a $\mathbb{Z}_2$-graded simple current extension of $V^{\langle\tau_e\rangle}$ which is isomorphic to $V = V^{\langle\tau_e\rangle} \oplus V_e(1/16)$.*

Proof: If $T_e(1/16)$ is an irreducible $T_e(0)$-module, then by the previous proposition the σ_e-conjugate $V_e(0) \oplus V_e(1/2)$-module of $V_e(1/16) = L(1/2, 1/16) \otimes T_e(1/16)$ is isomorphic to $W^1 = L(1/2, 1/16) \otimes Q_e(1/16)$. Then as the σ_e-conjugate extension of $V^{\langle\tau_e\rangle} \oplus V_e(1/16)$, $V^{\langle\tau_e\rangle} \oplus W^1$ has a structure of a $\mathbb{Z}_2$-graded extension. ∎

Remark 3.13. The above corollary implies that the $\mathbb{Z}_2$-twisted orbifold construction applied to V in the case of $\mathbb{Z}_2 = \langle\tau_e\rangle$ yields again V itself.

Theorem 3.14. *Under Hypothesis 1,*
(i) ψ_2 is surjective, that is, $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \simeq \langle\sigma_e\rangle.\mathrm{Aut}(T_e(0))$.
(ii) $\mathrm{Aut}(T_e(0) \oplus T_e(1/2)) \simeq 2.(C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)/\langle\sigma_e\rangle)$, *where* 2 *denotes the canonical $\mathbb{Z}_2$-symmetry on the SVOA $T_e(0) \oplus T_e(1/2)$.*
(iii) $|C_{(\mathrm{Aut}(V^{\langle\tau_e\rangle}))}(e) : C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle| \le 2$.
(iv) If $C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ is simple or has an odd order, then extensions in (i) and (ii) split. That is, $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e) \simeq \langle\sigma_e\rangle \times C_{\mathrm{Aut}(V)}(e)/\langle\tau_e\rangle$ and $\mathrm{Aut}(T_e(0) \oplus T_e(1/2)) \simeq 2 \times \mathrm{Aut}(T_e(0))$.

Proof: We have an injection from $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)/\langle\sigma_e\rangle$ to $\mathrm{Aut}(T_e(0))$ by Lemma 3.9. We show that every element in $\mathrm{Aut}(T_e(0))$ lifts to be an element in $C_{\mathrm{Aut}(V^{\langle\tau_e\rangle})}(e)$. By Proposition 3.11, every irreducible $T_e(0)$-module appears in one of $T_e(0)$, $T_e(1/2)$, $T_e(1/16)$

or $Q_e(1/16)$ as a submodule. In particular, we find that $T_e(0)$ is the only irreducible $T_e(0)$-module whose top weight is integral and $T_e(1/2)$ is the only irreducible $T_e(0)$-module whose top weight is in $1/2+\mathbb{N}$. Let $\rho \in \mathrm{Aut}(T_e(0))$. Then by considering top weights we can immediately see that $T_e(0)^\rho \simeq T_e(0)$ and $T_e(1/2)^\rho \simeq T_e(1/2)$. Then by Theorem 2.16 we have a lifting $\tilde{\rho} \in \mathrm{Aut}(T_e(0)\oplus T_e(1/2))$ such that $\tilde{\rho}T_e(0) = T_e(0)$, $\tilde{\rho}T_e(1/2) = T_e(1/2)$ and $\tilde{\rho}|_{T_e(0)} = \rho$. Since this lifting is unique up to a multiple of the canonical $\mathbb{Z}_2$-symmetry on $T_e(0)\oplus T_e(1/2)$, we have $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) \simeq 2.\mathrm{Aut}(T_e(0))$. Now consider the canonical extension of $\tilde{\rho}$ to $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$. We define $\tilde{\tilde{\rho}} \in C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$ by

$$\tilde{\tilde{\rho}}|_{L(1/2,h)\otimes T_e(h)} = \mathrm{id}_{L(1/2,h)} \otimes \tilde{\rho}$$

for $h = 0, 1/2$. Then by this lifting $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$ contains a subgroup to isomorphic to $2.\mathrm{Aut}(T_e(0))$. Moreover, the canonical $\mathbb{Z}_2$-symmetry on the SVOA $T_e(0)\oplus T_e(1/2)$ is naturally extended to $\sigma_e \in C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$. Clearly $\psi_2(\tilde{\tilde{\rho}}) = \rho$ and so ψ_2 is surjective. Hence we have the desired isomorphisms $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e) \simeq \langle \sigma_e\rangle.\mathrm{Aut}(T_e(0))$ and $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) \simeq 2.(C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)/\langle \sigma_e\rangle)$. This completes the proof of (i) and (ii).

Consider (iii). By Theorem 3.10, there are exactly three irreducible $V^{\langle \tau_e\rangle}$-modules whose top weights are integral, namely, $V^{\langle \tau_e\rangle}$, $V_e(1/16)$ and W^1. Since $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$ acts on the 2-point set $\{V_e(1/16), W^1\}$ as a permutation, there is a subgroup H of $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$ of index at most 2 such that $V_e(1/16)^\pi \simeq V_e(1/16)$ as a $V^{\langle \tau_e\rangle}$-module for all $\pi \in H$. Then there is a lifting $\tilde{\pi} \in C_{\mathrm{Aut}(V)}(e)$ of π such that $\psi_1(\tilde{\pi}) = \pi$ for each $\pi \in H$ by Theorem 2.16. Thus $|C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e) : C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle| \le 2$ and (iii) holds.

Consider (iv). Suppose that $C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle$ is a simple group or an odd group. Then $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$ contains a simple group $C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle$ with index at most 2 by (iii). However, since $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$ contains a normal subgroup $\langle \sigma_e\rangle$ of order 2, the index $|C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e) : C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle|$ must be 2 and hence we obtain the desired isomorphism $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e) = \langle \sigma_e\rangle \times C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle$. In this case, it is easy to see that the extension $\mathrm{Aut}(T_e(0)\oplus T_e(1/2)) = 2.\mathrm{Aut}(T_e(0))$ splits. ∎

Corollary 3.15. *If $C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle$ is simple, then $V_e(1/16)$ is an irreducible $V_e(0)$-module and $T_e(1/16)$ is an irreducible $T_e(0)$-module. Therefore, $V^{\langle \tau_e\rangle}\oplus W^1$ forms the σ_e-conjugate extension of $V = V^{\langle \tau_e\rangle}\oplus V_e(1/16)$ and is isomorphic to V.*

Proof: Let H be the subgroup of $C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e)$ which fixes $V_e(1/16)$ in the action on the 2-point set $\{V_e(1/16), W^1\}$. It is shown in the proof of (iii) of Theorem 3.14 that we have inclusions

$$H \subset C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle \subset C_{\mathrm{Aut}(V^{\langle \tau_e\rangle})}(e) = \langle \sigma_e\rangle \times C_{\mathrm{Aut}(V)}(e)/\langle \tau_e\rangle.$$

Therefore, $\sigma_e \notin H$ and hence the σ_e permutes $V_e(1/16)$ and W^1. Then $V_e(1/16)$ is an irreducible $V_e(0)$-module by Proposition 3.11 and hence $T_e(1/16)$ as a $T_e(0)$-module is irreducible. The rest of the assertion is now clear. ∎

Remark 3.16. A result similar to the assertion (iii) of Theorem 3.14 is already established in [M6]. Also, we should note that the idea of the above proof is already developed in [Sh].

4 2A-framed VOA

Definition 4.1. A simple vertex operator algebra (V, ω) is called *2A-framed* if there is an orthogonal decomposition $\omega = e^1 + \cdots + e^n$ such that each e^i generates a sub VOA isomorphic to $L(1/2, 0)$. The decomposition $\omega = e^1 + \cdots + e^n$ is called a *2A-frame* of V.

Remark 4.2. As shown in [DMZ], the Leech lattice VOA V_Λ and the moonshine VOA $V^\natural$ are examples of 2A-framed VOAs.

Let (V, ω) be a 2A-framed VOA with a 2A-frame $\omega = e^1 + \cdots + e^n$. Set $T := \mathrm{Vir}(e^1) \otimes \cdots \otimes \mathrm{Vir}(e^n)$, where $\mathrm{Vir}(e^i)$ denotes the sub VOA generated by e^i. Then $T \simeq L(1/2, 0)^{\otimes n}$ and V is a direct sum of irreducible T-submodules $\otimes_{i=1}^n L(1/2, h_i)$ with $h_i \in \{0, 1/2, 1/16\}$. For each irreducible T-module $\otimes_{i=1}^n L(1/2, h_i)$, we associate its 1/16-word $(\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_2^n$ by the rule $\alpha_i = 1$ if and only if $h_i = 1/16$. For each $\alpha \in \mathbb{Z}_2^n$, denote by V^α the sum of all irreducible T-submodules whose 1/16-words are equal to α and define a linear code $S \subset \mathbb{Z}_2^n$ by $S = \{\alpha \in \mathbb{Z}_2^n \mid V^\alpha \neq 0\}$. Then we have the *1/16-word decomposition* $V = \oplus_{\alpha \in S} V^\alpha$ of V. By the fusion rules (3.1), we have an S-graded structure $V^\alpha \cdot V^\beta \subset V^{\alpha+\beta}$. Namely, the dual group S^* of an abelian 2-group S acts on V, and we find that this automorphism group coincides with the elementary abelian 2-group generated by the first Miyamoto involutions $\{\tau_{e^i} \mid 1 \leq i \leq n\}$. Therefore, all V^α, $\alpha \in S$, are irreducible $V^{S^*} = V^0$-modules by [DM1]. Since there is no $L(1/2, 1/16)$-component in V^0, the fixed point subalgebra $V^{S^*} = V^0$ has the following shape:

$$V^0 = \bigoplus_{h_i \in \{0, 1/2\}} m_{h_1, \ldots, h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),$$

where $m_{h_1, \ldots, h_n}$ denotes the multiplicity. On V^0 we can define the second Miyamoto involutions σ_{e^i} for $i = 1, \ldots, n$. Denote by Q the elementary abelian 2-subgroup of $\mathrm{Aut}(V^0)$ generated by $\{\sigma_{e^i} \mid 1 \leq i \leq n\}$. Then by the quantum Galois theory [DM1] we have $(V^0)^Q = T$ and each $m_{h_1, \cdots, h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ is an irreducible T-submodule. Thus $m_{h_1, \cdots, h_n} \in \{0, 1\}$ and we obtain an even linear code $D := \{(2h_1, \cdots, 2h_n) \in \mathbb{Z}_2^n \mid m_{h_1, \cdots, h_n} \neq 0\}$ such that

$$V^0 = \bigoplus_{\alpha = (\alpha_1, \cdots, \alpha_n) \in D} L(1/2, \alpha_1/2) \otimes \cdots \otimes L(1/2, \alpha_n/2). \tag{4.1}$$

We call a pair (D, S) the *structure codes* of a 2A-framed VOA V. Since powers of z in an $L(1/2, 0)$-intertwining operator of type $L(1/2, 1/2) \times L(1/2, 1/2) \to L(1/2, 1/16)$ are

half-integral, structure codes satisfy $D \subset S^{\perp}$. Since all V^{α}, $\alpha \in S$, are irreducible modules for V^0, the representation theory of V^0 is important to study a 2A-framed VOA V. We review Miyamoto's results on the code VOAs [M3] [M4] in the next subsection.

4.1 Code VOA

Below we often identify the code $\mathbb{Z}_2^n$ with the power set of an n-point set $\Omega = \{1, 2, \dots, n\}$ with the symmetric difference operation. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_2^n$, we denote by $\mathrm{Supp}(\alpha)$ the subset $\{i \mid \alpha_i \neq 0\}$ of Ω. Let D be a subcode of $\mathbb{Z}_2^n$. Set $D^{(0)} := \{\alpha \in D \mid \langle \alpha, \alpha \rangle = 0\}$ and $D^{(1)} := \{\alpha \in D \mid \langle \alpha, \alpha \rangle = 1\}$. For a code-word $\alpha = (\alpha_1, \dots, \alpha_n) \in D$, we define

$$U^{\alpha} := L(1/2, \alpha_1/2) \otimes \cdots \otimes L(1/2, \alpha_n/2).$$

Theorem 4.3. *([M2]) For any linear code $D \subset \mathbb{Z}_2^n$, there exists a unique simple vertex operator superalgebra structure on $U_D := \oplus_{\alpha \in D} U^{\alpha}$ as an extension of $U^0 = L(1/2, 0)^{\otimes n}$. The even part is $U_{D^{(0)}} := \oplus_{\alpha \in D^{(0)}} U^{\alpha}$ and is a $D^{(0)}$-graded simple current extension of U^0, and the odd part is $U_{D^{(1)}} := \oplus_{\beta \in D^{(1)}} U^{\beta}$.*

Let D be an even subcode of $\mathbb{Z}_2^n$. The simple VOA U_D defined in the theorem above is called the *code VOA associated to a code D*. The representation theory of U_D is deeply studied in [M3]. We recall some results from [M3]. Since U_D is a D-graded simple current extension of a rational VOA $U^0 = L(1/2, 0)^{\otimes n}$, it is also rational. Let M be an irreducible U_D-module. Take an irreducible U^0-submodule W of M. Then W is isomorphic to $\otimes_{i=1}^n L(1/2, h_i)$ with $h_i \in \{0, 1/2, 1/16\}$. Define the 1/16-word $\tau(W) = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_2^n$ of W by $\alpha_i = 1$ if and only if $h_i = 1/16$. Then by the fusion rules (3.1), $\tau(W)$ is determined independently of a choice of W and hence we can define the 1/16-word $\tau(M) \in \mathbb{Z}_2^n$ of M by $\tau(M) := \tau(W)$.

Assume that $\tau(M) = (0^n)$. Then W is isomorphic to $\otimes_{i=1}^n L(1/2, h_i)$ with $h_i \in \{0, 1/2\}$. We set $\gamma := (2h_1, \dots, 2h_n) \in \mathbb{Z}_2^n$. Since $L(1/2, 0)$ is a simple current $L(1/2, 0)$-module, we have $D_W = 0$ and hence M is uniquely determined by W by Theorem 2.11 and has a shape

$$U_{D+\gamma} := \oplus_{\alpha \in D+\gamma} U^{\alpha}.$$

We call an irreducible U_D-module $U_{D+\gamma}$ a *coset module*. By Theorem 2.13 and the fusion rules (3.1), we have the following fusion rules for $\gamma_1, \gamma_2 \in \mathbb{Z}_2^n$:

$$U_{D+\gamma_1} \times U_{D+\gamma_2} = U_{D+\gamma_3}. \tag{4.2}$$

In particular, $U_{D+\delta} \times U_{D+\delta} = U_D$ for any $\delta \in \mathbb{Z}_2^n$ and hence all the coset modules are simple current U_D-modules by Lemma 2.6.

4.2 The Hamming code VOA

Let H_8 be the $[8,4,4]$-Hamming code:

$$H_8 := \mathrm{Span}_{\mathbb{Z}_2}\{(11111111), (11110000), (11001100), (10101010)\}.$$

It is well-known that H_8 is the unique doubly even self-dual linear code of length 8 up to isomorphism. Let us consider the Hamming code VOA U_{H_8}. In order to construct 2A-framed VOAs, we will need some special properties that the Hamming code VOA U_{H_8} possesses. Roughly speaking, we can identify $L(1/2,1/16)$ with $L(1/2,0)$ and $L(1/2,1/2)$ by the symmetry of the Hamming code VOA.

Let X be an irreducible U_{H_8}-module whose top weight is in $\frac{1}{2}\mathbb{N}$. Then $\tau(X)=(0^8)$ or (1^8) and if $\tau(X)=(0^8)$ then $X\simeq U_{H_8+\gamma}$ for some $\gamma\in\mathbb{Z}_2^8$. If $\tau(X)=(1^8)$, then it is shown in [M3] that there is a unique linear character χ on H_8 such that

$$X \simeq \mathrm{Ind}_{U^0}^{U_{H_8}}(L(1/2,1/16)^{\otimes 8},\chi) = L(1/2,1/16)^{\otimes 8}\otimes_{\mathbb{C}} v_\chi, \tag{4.3}$$

where U^α, $\alpha\in H_8$, acts on $L(1/2,16)^{\otimes 8}$ by the fusion rule (3.1) and on $\mathbb{C}v_\chi$ as a scalar $\chi(\alpha)$ according to the character χ. Since the dual group H_8^* of H_8 is naturally isomorphic to $\mathbb{Z}_2^8/H_8$, we can find a unique coset $\delta_\chi+H_8\in\mathbb{Z}_2^8/H_8$ such that $\chi(\alpha)=\langle\delta_\chi,\alpha\rangle$ for all $\alpha\in H_8$. So in the following we regard χ as an element in $\mathbb{Z}_2^8$. Set $H(1/16,\chi)=L(1/2,1/16)^{\otimes 8}\otimes_{\mathbb{C}} v_\chi$ for $\chi\in\mathbb{Z}_2^8$. Then $H(1/16,\chi_1)\simeq H(1/16,\chi_2)$ as U_{H_8}-modules if and only if $\chi_1-\chi_2\in H_8$ and the set of inequivalent irreducible U_{H_8}-modules whose top weights are contained in $\frac{1}{2}\mathbb{N}$ is given by

$$\{U_{H_8+\gamma},\ H(1/16,\chi) \mid \gamma+H_8, \chi+H_8\in\mathbb{Z}_2^8/H_8\}.$$

Surprisingly, we can identify a non-simple current $L(1/2,0)^{\otimes 8}$-module $L(1/2,1/16)^{\otimes 8}$ with a coset module as follows:

Theorem 4.4. *([M4]) For each $\chi\in\mathbb{Z}_2^8$, there is an automorphism $\sigma\in\mathrm{Aut}(U_{H_8})$ such that the σ-conjugate module $H(1/16,\chi)^\sigma$ is isomorphic to a coset module $U_{H_8+\gamma}$ for some $\gamma\in\mathbb{Z}_2^8$ with $\langle\gamma,\gamma\rangle=1$. In particular, $H(1/16,\chi)$ is a simple current U_{H_8}-module.*

Corollary 4.5. *([M4]) As a $\mathbb{Z}_2$-graded simple current extension of U_{H_8}, there is a unique simple SVOA structure on $U_{H_8}\oplus H(1/16,\chi)$ for all $\chi\in\mathbb{Z}_2^8$.*

Proof: We can take an irreducible coset U_{H_8}-module $U_{H_8+\gamma}$ with $\langle\gamma,\gamma\rangle=1$ such that there is an automorphism $\sigma\in\mathrm{Aut}(U_{H_8})$ such that the conjugate module $(U_{H_8+\gamma})^\sigma$ is isomorphic to $H(1/16,\chi)$ by Theorem 4.4. Then $U_{H_8}\oplus U_{H_8+\gamma}$ and $U_{H_8}\oplus H(1/16,\chi)$ form mutually conjugate $\mathbb{Z}_2$-graded simple current extensions of U_{H_8} under $\sigma\in\mathrm{Aut}(U_{H_8})$. Since $H_8\cup(H_8+\gamma)$ is an odd code, $U_{H_8}\oplus U_{H_8+\gamma}$ is a simple SVOA. Then so is $U_{H_8}\oplus H(1/16,\chi)$. ∎

As an application of Proposition 4.4, the following fusion rules are established in [M4]:

Theorem 4.6. *([M4]) We have the following fusion rules:*

$$\begin{aligned} U_{H_8+\alpha} \times U_{H_8+\beta} &= U_{H_8+\alpha+\beta}, \\ U_{H_8+\alpha} \times H(1/16,\beta) &= H(1/16,\alpha+\beta), \\ H(1/16,\alpha) \times H(1/16,\beta) &= U_{H_8+\alpha+\beta}, \end{aligned}$$

where $\alpha,\beta \in \mathbb{Z}_2^8$.

Thanks to Corollary 4.5 and Theorem 4.6, if an even linear code D contains many subcodes isomorphic to the Hamming code H_8, then we can construct simple current extensions of the code VOA U_D by using Theorem 2.17.

4.3 Construction of 2A-framed VOA

In this subsection we give a refinement Miyamoto's construction of 2A-framed VOAs in [M4]. Here we assume the following:

Hypothesis 2.
(1) (D,S) is a pair of even linear even codes of $\mathbb{Z}_2^n$ such that

(1-i) $D \subset S^\perp$,

(1-ii) for each $\alpha \in S$, there is a subcode $E^\alpha \subset D$ such that E^α is a direct sum of the Hamming code H_8 and $\mathrm{Supp}(E^\alpha) = \mathrm{Supp}(\alpha)$, where $\mathrm{Supp}(A)$ denotes $\cup_{\beta\in A}\mathrm{Supp}(\beta)$ for a subset A of $\mathbb{Z}_2^n$.

(2) $V^0 = U_D$ is the code VOA associated to the code D.

(3) $\{V^\alpha \mid \alpha \in S\}$ is a set of irreducible V^0-modules such that

(3-i) $\tau(V^\alpha) = \alpha$ for all $\alpha \in S$,

(3-ii) all V^α, $\alpha \in S$, have integral top weights,

(3-iii) the fusion product $V^\alpha \boxtimes_{V^0} V^\beta$ contains at least one $V^{\alpha+\beta}$. That is, there is a non-trivial V^0-intertwining operator of type $V^\alpha \times V^\beta \to V^{\alpha+\beta}$ for any $\alpha,\beta \in S$.

Under Hypothesis 2 we will prove that $V := \oplus_{\alpha\in S} V^\alpha$ has a structure of an S-graded simple current extension of V^0. Before we begin the proof, we prepare some lemmas.

Lemma 4.7. *Under Hypothesis 2, all V^α, $\alpha \in S$, are simple current V^0-modules and we have the fusion rules $V^\alpha \times V^\beta = V^{\alpha+\beta}$ of V^0-modules for all $\alpha,\beta \in S$.*

Proof: Suppose the fusion rule $V^\alpha \times V^\alpha = V^0$ of V^0-modules holds. Then by Lemma 2.6, V^α is a simple current V^0-module because $V^0 = U_D$ is a rational C_2-cofinite VOA of CFT-type. Then by Hypothesis 2 (3-iii) we have the desired fusion rule $V^\alpha \times V^\beta = V^{\alpha+\beta}$. Therefore, we only prove the fusion rule $V^\alpha \times V^\alpha = V^0$ for each $\alpha \in S$. By Hypothesis 2

(1-i), D contains a subcode E^α which is isomorphic to a direct sum of H_8 and $\mathrm{Supp}(E^\alpha) = \mathrm{Supp}(\alpha)$. We may assume that $\alpha = (1^{8s}0^t)$ with $8s + t = n$. Then U_D contains a sub VOA

$$L := U_{E^\alpha} \otimes L(1/2,0)^{\otimes t} \simeq (U_{H_8})^{\otimes s} \otimes L(1/2,0)^{\otimes t}$$

and V^α contains an irreducible L-submodule X isomorphic to

$$H(1/16,\chi_1) \otimes \cdots H(1/16,\chi_s) \otimes L(1/2,h_1) \otimes \cdots \otimes L(1/2,h_t)$$

with $\chi_i \in \mathbb{Z}_2^8$, $1 \le i \le s$, and $h_j \in \{0, 1/2\}$, $1 \le j \le t$. Let $D = \sqcup_{i=0}^k (E^\alpha + \beta_i)$ be a coset decomposition. We write $\beta_i = \gamma_i + \delta_i$ such that $\mathrm{Supp}(\gamma_i) \subset \mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\delta_i) \cap \mathrm{Supp}(\alpha) = \emptyset$. Then $U_{E^\alpha+\beta_i}$ is isomorphic to

$$U_{E^\alpha+\gamma_i} \otimes L(1/2,(\delta_i)_{8s+1}/2) \otimes \cdots \otimes L(1/2,(\delta_i)_n/2)$$

as an L-module and $U_D = \oplus_{i=1}^k U_{E^\alpha+\beta_i}$ is a D/E^α-graded simple current extension of L. Then by the fusion rules (3.1) and Theorem 4.6, $(U_{E^\alpha+\beta_i}) \boxtimes_L X$ is an irreducible L-module and $(U_{E^\alpha+\beta_i}) \boxtimes_L X \not\simeq (U_{E^\alpha+\beta_j}) \boxtimes_L X$ unless $i = j$. Therefore, V^α as an L-module is isomorphic to $V^\alpha = \oplus_{i=1}^k (V_{E^\alpha+\beta_i}) \boxtimes_L X$. Namely, V^α is a D/E^α-stable U_D-module. Then by Theorem 2.13 together with fusion rules (3.1) and Theorem 4.6 we have a fusion rule $V^\alpha \times V^\alpha = V^0$ of U_D-modules which is a lifting of the fusion rule $X \times X = L$ of L-modules. ∎

Lemma 4.8. *Under Hypothesis 2, the space $V^0 \oplus V^\alpha$ forms a simple VOA as a $\mathbb{Z}_2$-graded simple current extension of V^0 for each $\alpha \in S \setminus 0$.*

Proof: Here we use the same notation as in the proof of Lemma 4.7. By the coset decomposition $D = \sqcup_{i=1}^k (E^\alpha + \beta_i)$, $V^0 = U_D = \oplus_{i=1}^k U_{E^\alpha+\beta_i}$ is a D/E^α-graded simple current extension of $L = U_{E^\alpha} \otimes L(1/2,0)^{\otimes t} \simeq (U_{H_8})^{\otimes s} \otimes L(1/2,0)^{\otimes t}$. By the fusion rule $X \times X = L$ of L-modules, X is a simple current L-module by Lemma 2.6. Then by the associativity of fusion products (cf. [H4]), $(U_{E^\alpha+\beta_i}) \boxtimes_L X$ is also a simple current L-module. Thus we obtain the set of inequivalent simple current L-modules

$$\mathcal{S} = \{\, U_{E^\alpha+\beta_i},\ (U_{E^\alpha+\beta_j}) \boxtimes_L X \mid 1 \le i, j \le k \,\}$$

with the following $((D/E^\alpha) \oplus \mathbb{Z}_2)$-graded fusion rules:

$$\begin{aligned}
&U_{E^\alpha+\beta_i} \times U_{E^\alpha+\beta_j} = U_{E^\alpha+\beta_i+\beta_j},\\
&U_{E^\alpha+\beta_i} \times (U_{E^\alpha+\beta_j} \boxtimes_L X) = (U_{E^\alpha+\beta_i+\beta_j}) \boxtimes_L X,\\
&(U_{E^\alpha+\beta_i} \boxtimes_L X) \times (U_{E^\alpha+\beta_j} \boxtimes_L X) = U_{E^\alpha+\beta_i+\beta_j}.
\end{aligned}$$

Since $U_D = \oplus_{i=1}^k U_{E^\alpha+\beta_i}$ has a structure of a D/E^α-graded simple current extension of L and $L \oplus X$ has a structure of a $\mathbb{Z}_2$-graded simple current extension of L by Corollary 4.5,

we can apply Theorem 2.17 to $\mathcal{S}$ and hence we obtain a $((D/E^\alpha)\oplus\mathbb{Z}_2)$-graded simple current extension

$$\{\oplus_{i=1}^k U_{E^\alpha+\beta_i}\}\bigoplus\{\oplus_{i=1}^k(U_{E^\alpha+\beta_i})\boxtimes_L X\}$$

of L. Since $V^0=\oplus_{i=1}^k U_{E^\alpha+\beta_i}$ and $V^\alpha=\oplus_{i=1}^k(U_{E^\alpha+\beta_i})\boxtimes_L X$, the $\mathbb{Z}_2$-graded space $V^0\oplus V^\alpha$ carries a simple VOA structure which is the desired $\mathbb{Z}_2$-graded simple current extension of V^0. ∎

Now we can prove

Theorem 4.9. *([M4]) Under Hypothesis 2, the space $V=\oplus_{\alpha\in S}V^\alpha$ has a unique structure of a simple VOA as an S-graded simple current extension of V^0. In particular, there exists a 2A-framed VOA whose structure codes are (D,S).*

Proof: Let $\{\alpha_1,\dots,\alpha_r\}$ be a linear basis of S and set $S^i:=\mathrm{Span}_{\mathbb{Z}_2}\{\alpha_1,\dots,\alpha_i\}$ for $1\le i\le r$. We proceed by induction on r. The case $r=0$ is trivial and the case $r=1$ is given by Lemma 4.8. Now assume that $\oplus_{\beta\in S^i}V^\beta$ has a structure of a simple VOA for $1\le i\le r-1$. Then the set

$$\mathcal{T}=\{\,V^\beta,\ V^{\beta+\alpha_{i+1}}\mid\beta\in S^i\,\}$$

consists of inequivalent simple current V^0-modules with $(S^i\oplus\mathbb{Z}_2)=S^{i+1}$-graded fusion rules:

$$V^{\beta_1}\times V^{\beta_2}=V^{\beta_1+\beta_2},\quad V^{\beta_1}\times V^{\beta_2+\alpha_{i+1}}=V^{\beta_1+\beta_2+\alpha_{i+1}},\quad V^{\beta_1+\alpha_{i+1}}\times V^{\beta_2+\alpha_{i+1}}=V^{\beta_1+\beta_2},$$

where $\beta_1,\beta_2\in S^i$. By inductive assumption, $\oplus_{\beta\in S^i}V^\beta$ is an S^i-graded simple current extension of V^0, and by Lemma 4.8, a direct sum $V^0\oplus V^{\alpha_{i+1}}$ becomes a $\mathbb{Z}_2$-graded simple current extension of V^0. Therefore, we can apply Theorem 2.17 to $\mathcal{T}$ to obtain the S^{i+1}-graded simple current extension $\oplus_{\beta\in S^{i+1}}V^\alpha$ of V^0. Repeating this procedure, we finally obtain $S^r=S$-graded simple current extension $V=\oplus_{\alpha\in S}V^\alpha$ of $V^0=U_D$. ∎

Remark 4.10. In [M4], Miyamoto assumed stronger conditions than that in Hypothesis 2. In particular, he assumed that the structure codes (D,S) are of length $8k$ for some positive integer k. Our refinement enable us to construct 2A-framed VOAs with structure codes of any length as long as Hypothesis 2 is satisfied.

Extension to SVOA. Let V be a 2A-framed VOA with structure codes (D,S). Then by definition we have a decomposition $V=\oplus_{\alpha\in S}V^\alpha$ such that $V^0\simeq U_D$ and $\tau(V^\alpha)=\alpha$. Assume that the pair (D,S) satisfies the condition (1) of Hypothesis 2. Then V is an S-graded simple current extension of V^0 by Lemma 4.7. Suppose that there is a vector $\gamma\in S^\perp\setminus D$ such that $\langle\gamma,\gamma\rangle=1$. Since the powers of z in an $L(1/2,0)$-intertwining

operator of type $L(1/2,1/2)\times L(1/2,1/16)\to L(1/2,1/16)$ are half-integral, the powers of z in a U_D-intertwining operator of type $U_{D+\gamma}\times V^\alpha\to U_{D+\gamma}\boxtimes_{U_D} V^\alpha$ are integral for all $\alpha\in S$. Therefore, all $V^\alpha\boxtimes_{U_D} U_{D+\gamma}$, $\alpha\in S$, have half-integral top weights. Since the $1/16$-word of $V^\alpha\boxtimes_{U_D} U_{D+\gamma}$ is α by the fusion rules (3.1), the induced module

$$\mathrm{Ind}_{V^0}^{V} U_{D+\gamma} = \bigoplus_{\alpha\in S} V^\alpha \boxtimes_{V^0} U_{D+\gamma}$$

has a unique V-module structure by Theorem 2.11. Since $U_D\oplus U_{D+\gamma}$ is a simple current super-extension of $V^0=U_D$ by Theorem 4.3, by applying Theorem 2.17 we obtain:

Theorem 4.11. *With reference to the setup above, $V\oplus \mathrm{Ind}_{V^0}^{V} U_{D+\gamma}$ forms a simple current super-extension of V.*

5 The baby-monster SVOA

As shown in [DMZ], the moonshine VOA $V^\natural$ has a 2A-frame $\omega^\natural = e^1+\cdots+e^{48}$. One of its structure codes are determined in [DGH] and [M5]. Let S be the Reed-Müller code $RM(4,1)$ defined as follows:

$$S=\mathrm{Span}_{\mathbb{Z}_2}\{(1^{16}),\ (1^8 0^8),\ (1^4 0^4 1^4 0^4),\ (1^2 0^2 1^2 0^2 1^2 0^2 1^2 0^2),\ (1010101010101010)\}\subset \mathbb{Z}_2^{16}.$$

Then define

$$S^\natural := \{(\alpha,\alpha,\alpha),\ (\alpha^c,\alpha,\alpha),\ (\alpha,\alpha^c,\alpha),\ (\alpha,\alpha,\alpha^c)\in \mathbb{Z}_2^{48}\mid \alpha\in S,\ \alpha^c:=\alpha+(1^{16})\}$$

and $D^\natural := (S^\natural)^\perp$.

Theorem 5.1. *([DGH] [M5]) The moonshine VOA $V^\natural$ has a structure codes $(D^\natural,S^\natural)$.*

One can easily check that the pair $(D^\natural,S^\natural)$ satisfies the condition (1) of Hypothesis 2. Thus, by Lemma 4.7, we have

Corollary 5.2. *Let $V^\natural=\oplus_{\alpha\in S^\natural}(V^\natural)^\alpha$ be the 1/16-word decomposition according to the structure codes $(D^\natural,S^\natural)$. Then the pair $(D^\natural,S^\natural)$ and the set $\{(V^\natural)^\alpha\mid \alpha\in S^\natural\}$ satisfy Hypothesis 2. Therefore, $V^\natural$ is an $S^\natural$-graded simple current extension of $(V^\natural)^0=U_{D^\natural}$.*

Now set $e=e^1$ and consider the commutant subalgebra $T_e^\natural(0)$ of $\mathrm{Vir}(e)$ in $V^\natural$. Since $\{1\}\cap \mathrm{Supp}(S^\natural)\neq\emptyset$, $V_e^\natural(1/16)$ is not zero. Then by the condition (1) of Hypothesis 2, $V_e(1/2)$ is not zero, too. Therefore, we obtain a decomposition

$$V^\natural = L(1/2,0)\otimes T_e^\natural(0)\oplus L(1/2,1/2)\otimes T_e^\natural(1/2)\oplus L(1/2,1/16)\otimes T_e^\natural(1/16)$$

such that $T_e^\natural(h) \neq 0$ for $h = 0, 1/2, 1/16$. By Theorem 3.6, we know that $T_e^\natural(0) \oplus T_e^\natural(1/2)$ has a structure of a simple vertex operator superalgebra. This algebra was first considered by Höhn [H1] and he called it the *baby-monster SVOA*, because the centralizer $C_{\mathrm{Aut}(V^\natural)}(\tau_e)$ is isomorphic to the 2-fold central extension $\langle \tau_e \rangle \cdot \mathbb{B}$ of the baby-monster sporadic finite simple group $\mathbb{B}$ [ATLAS] and so $\mathbb{B}$ naturally acts on it. Following him, we set $V\!B^0 := T_e^\natural(0)$, $V\!B^1 := T_e^\natural(1/2)$ and $V\!B := T_e^\natural(0) \oplus T_e^\natural(1/2)$. We also know that $T_e^\natural(1/16)$ is an irreducible $\mathbb{Z}_2$-twisted $V\!B$-module by Theorem 3.8, and so we set $V\!B_T := T_e^\natural(1/16)$ for convention. Since all the conformal vectors of $V^\natural$ with central charge $1/2$ are conjugate under the Monster $\mathbb{M} = \mathrm{Aut}(V^\natural)$ by [C] and [M1], the algebraic structures on $V\!B$ and $V\!B_T$ are independent of choice a conformal vector $e = e^1$.

By definition, the Virasoro vector of $V\!B^0$ is given by $\omega^\natural - e^1 = e^2 + \cdots + e^{48}$. Thus $V\!B^0$ is a 2A-framed VOA. We compute the structure codes of $V\!B^0$. Set

$$(S^\natural)^0 := \{\alpha \in S^\natural \mid \{1\} \cap \mathrm{Supp}(\alpha) = \emptyset\}, \quad (S^\natural)^1 := \{\alpha \in S^\natural \mid \{1\} \cap \mathrm{Supp}(\alpha) = \{1\}\}.$$

Then $S^\natural = (S^\natural)^0 \sqcup (S^\natural)^1$ and by definition of Miyamoto involution, $(V^\natural)^{\langle \tau_e \rangle} = \oplus_{\alpha \in (S^\natural)^0} (V^\natural)^\alpha$ and $V_e^\natural(1/16) = \oplus_{\beta \in (S^\natural)^1} (V^\natural)^\beta$. Since $V^\natural$ is an $S^\natural$-graded simple current extension of $(V^\natural)^0$, the fixed point subalgebra $(V^\natural)^{\langle \tau_e \rangle}$ is also an $(S^\natural)^0$-graded simple current extension of $(V^\natural)^0$ and $V_e^\natural(1/16)$ is a simple current $(V^\natural)^{\langle \tau_e \rangle}$-module by Corollary 2.14. Now define $\phi_\epsilon : \mathbb{Z}_2^{47} \hookrightarrow \mathbb{Z}_2^{48}$ by $\mathbb{Z}_2^{47} \ni \alpha \mapsto (\epsilon, \alpha) \in \mathbb{Z}_2^{48}$ for $\epsilon = 0, 1$, and set

$$D^{\flat,\epsilon} := \{\alpha \in \mathbb{Z}_2^{47} \mid \phi_\epsilon(\alpha) \in D^\natural\}, \ \epsilon = 0, 1, \quad S^\flat := \{\beta \in \mathbb{Z}_2^{47} \mid \phi_0(\beta) \in (S^\natural)^0\}.$$

Proposition 5.3. *The structure codes of $V\!B^0$ with respect to the 2A-frame $e^2 + \cdots + e^{48}$ are $(D^{\flat,0}, S^\flat)$.*

Proof: For $\alpha \in (S^\natural)^0$, define $(V^\natural)^{\alpha,\epsilon}$ to be the sum of all irreducible $\otimes_{i=1}^{48} \mathrm{Vir}(e^i)$-submodules of $(V^\natural)^\alpha$ whose $\mathrm{Vir}(e^1)$-components are isomorphic to $L(1/2, \epsilon/2)$ for $\epsilon = 0, 1$. Then $V_e^\natural(1/2) \neq 0$ implies that $(V^\natural)^{\alpha,\epsilon} \neq 0$ for all $\alpha \in (S^\natural)^0$ and $\epsilon = 0, 1$. Therefore, $(V^\natural)^\alpha = (V^\natural)^{\alpha,0} \oplus (V^\natural)^{\alpha,1}$ and we obtain 1/16-word decompositions $V_e^\natural(0) = \oplus_{\alpha \in (S^\natural)^0} (V^\natural)^{\alpha,0}$ and $V_e^\natural(1/2) = \oplus_{\alpha \in (S^\natural)^0} (V^\natural)^{\alpha,1}$. Since $D^\natural = \phi_0(D^{\flat,0}) \sqcup \phi_1(D^{\flat,1})$, $(V^\natural)^{0,0} \simeq L(1/2, 0) \otimes U_{D^{\flat,0}}$. Thus $V\!B^0$ has a 1/16-word decomposition $V\!B^0 = \oplus_{\alpha \in S^\flat} (V\!B^0)^\alpha$ such that $\tau((V\!B^0)^\alpha) = \alpha$ and $(V\!B^0)^0 \simeq U_{D^{\flat,0}}$. Hence the structure codes of $V\!B^0$ are $(D^{\flat,0}, S^\flat)$. ∎

Remark 5.4. By the proof above, we find that $V\!B^1$ also has a 1/16-word decomposition $V\!B^1 = \oplus_{\alpha \in S^\flat} (V\!B^1)^\alpha$ such that $\tau((V\!B^1)^\alpha) = \alpha$. In particular, $(V\!B^1)^0$ is isomorphic to a coset module $U_{D^{\flat,1}}$.

The following is easy to see:

Lemma 5.5. *The pair $(D^{\flat,0}, S^\flat)$ satisfies the condition (1) of Hypothesis 2.*

Therefore, $V\!B^0 = \oplus_{\alpha \in S^\flat} (V\!B^0)^\alpha$ is an $S^\flat$-graded simple current extension of the code VOA $U_{D^{\flat,0}}$. Since the pair $(D^{\flat,0}, S^\flat)$ and the set $\{(V\!B^0)^\alpha \mid \alpha \in S^\flat\}$ satisfy Hypothesis 2, we can construct $V\!B^0$ without reference to $V^\natural$ by Theorem 4.9.

Proposition 5.6. *$V\!B^1$ is a simple current $V\!B^0$-module.*

Proof: By Remark 5.4, $V\!B^1$ is isomorphic to the induced module $\mathrm{Ind}_{U_{D^{\flat,0}}}^{V\!B^0} U_{D^{\flat,1}}$ which is an $S^\flat$-stable $V\!B^0$-module. Therefore, we have the fusion rule

$$V\!B^1 \times V\!B^1 = \mathrm{Ind}_{U_{D^{\flat,0}}}^{V\!B^0} (U_{D^{\flat,1}} \boxtimes_{U_{D^{\flat,0}}} U_{D^{\flat,1}}) = \mathrm{Ind}_{U_{D^{\flat,0}}}^{V\!B^0} U_{D^{\flat,0}} = V\!B^0$$

by Theorem 2.13. Thus $V\!B^1$ is a simple current $V\!B^0$-module by Lemma 2.6. ∎

Remark 5.7. We note that by using Theorem 4.11 we can define the SVOA structure on $V\!B$ without reference to $V^\natural$.

Up to now, we have established that $V^\natural$ and its conformal vector e satisfy all the conditions in Hypothesis 1. Moreover, it is shown in [C] and [M1] that there is a one-to-one correspondence between the set of conformal vectors of $V^\natural$ with central charge 1/2 and the set of corresponding Miyamoto involutions on $V^\natural$ which is known to be the 2A-conjugacy class of the Monster. Therefore, $C_{\mathrm{Aut}(V^\natural)}(e) = C_{\mathbb{M}}(\tau_e) \simeq \langle \tau_e \rangle \cdot \mathbb{B}$. Thus, $C_{\mathrm{Aut}(V^\natural)}(e)/\langle \tau_e \rangle$ is a simple group and we can apply Theorem 3.14 to $V\!B$.

Theorem 5.8. *(1) The SVOA $V\!B$ obtained from $V^\natural$ by cutting off the Ising model is a simple SVOA.*
(2) The piece $V\!B_T$ obtained from $V^\natural$ is an irreducible $\mathbb{Z}_2$-twisted $V\!B$-module.
(3) $\mathrm{Aut}(V\!B^0) \simeq \mathbb{B}$ *and* $\mathrm{Aut}(V\!B) \simeq 2 \times \mathbb{B}$.
(4) $V\!B_T$ as a $V\!B^0$-module is irreducible. Thus, there are exactly three irreducible $V\!B^0$-modules, $V\!B^0$, $V\!B^1$ and $V\!B_T$.
(5) The fusion rules for irreducible $V\!B^0$-modules are as follows:

$$V\!B^1 \times V\!B^1 = V\!B^0, \quad V\!B^1 \times V\!B_T = V\!B_T, \quad V\!B_T \times V\!B_T = V\!B^0 + V\!B^1.$$

Proof: (1) follows from Theorem 3.6, (2) follows from 3.8 and (3) will follow from Theorem 3.14 and the fact $C_{\mathrm{Aut}(V^\natural)}(e) = \langle \tau_e \rangle \cdot \mathbb{B}$.

Consider (4). By Corollary 3.15, $V\!B_T$ as a $V\!B^0$-module is irreducible. Then the assertion follows from Proposition 3.11.

Consider (5). We only have to show the fusion rule $V\!B_T \times V\!B_T = V\!B^0 + V\!B^1$. By considering the 1/16-word of $V\!B_T$, the fusion product $V\!B_T \times V\!B_T$ is contained in $\mathbb{N}V\!B^0 \oplus \mathbb{N}V\!B^1$ in the fusion algebra for $V\!B^0$. Write $V\!B_T \times V\!B_T = n_0 V\!B^0 + n_1 V\!B^1$ with $n_0, n_1 \in \mathbb{N}$. Then the simplicity of $V^\natural$ implies $n_0 \neq 0$ and $n_1 \neq 0$. And by applying $V\!B^1$ to $V\!B_T \times V\!B_T$, we see that $n_0 = n_1$. Since the dual module of $V\!B_T$ is isomorphic to $V\!B_T$, the space of $V\!B^0$-intertwining operator of type $V\!B_T \times V\!B_T \to V\!B^0$ is one-dimensional. Thus $n_0 = n_1 = 1$ as desired. ∎

Remark 5.9. The assertion (1) of Theorem 5.8 is already shown by Höhn in [Hö1], and (3) of Theorem 5.8 is also proved in [Hö2]. However, Höhn's proofs in [Hö1] [Hö2] and ours are quite different. In particular, in [Hö2], he used many results on the baby-monster simple group. Our argument can be applied to any 2A-framed VOAs satisfying Hypothesis 1 and Hypothesis 2 since we have only used the facts that $C_{\mathrm{Aut}(V^\natural)}(e) = C_{\mathrm{Aut}(V^\natural)}(\tau_e)$ and $C_{\mathrm{Aut}(V^\natural)}(\tau_e)/\langle\tau_e\rangle$ is a simple group.

The classification of irreducible $V\!B^0$-modules has many interesting corollaries.

Corollary 5.10. *The irreducible 2A-twisted $V^\natural$-module has a shape*

$$L(1/2,1/2)\otimes V\!B^0 \oplus L(1/2,0)\otimes V\!B^1 \oplus L(1/2,1/16)\otimes V\!B_T.$$

Proof: Follows from Theorem 5.8, Theorem 3.10 and Proposition 3.11. ∎

Remark 5.11. A straightforward construction of the 2A-twisted (and 2B-twisted) $V^\natural$-module is given by Lam [L2]. In his construction, it is given as $U_{D^\natural+\gamma} \boxtimes_{U_{D^\natural}} V^\natural$ with $\gamma = (10^{47}) \in (\mathbb{Z}/2\mathbb{Z})^{48}$.

Corollary 5.12. *For any conformal vector $e \in V^\natural$ with central charge 1/2, there is no automorphism ρ on $V^\natural$ such that $\rho(V_e^\natural(h)) = V_e^\natural(h)$ for $h = 0, 1/2$ and $\rho|_{(V^\natural)^{\langle\tau_e\rangle}} = \sigma_e$.*

Proof: Suppose such an automorphism ρ exists. We remark that ρ also preserves the space $V_e(1/16)$ as $\rho \in C_{\mathrm{Aut}(V)}(e)$. We view $V_e^\natural(1/16)$ as a $(V^\natural)^{\langle\tau_e\rangle}$-module by a restriction of the vertex operator map $Y_{V^\natural}(\cdot,z)$ on $V^\natural$. Consider the σ_e-conjugate $(V^\natural)^{\langle\tau_e\rangle}$-module $V_e^\natural(1/16)^{\sigma_e}$. By Theorem 5.8 and Proposition 3.11, $V_e^\natural(1/16)^{\sigma_e}$ is not isomorphic to $V_e^\natural(1/16)$ as a $(V^\natural)^{\langle\tau_e\rangle}$-module. On the other hand, we can take a canonical linear isomorphism $\varphi : V_e^\natural(1/16) \to V_e^\natural(1/16)^{\sigma_e}$ such that $Y_{V_e^\natural(1/16)^{\sigma_e}}(a,z)\varphi v = \varphi Y_{V^\natural}(\sigma_e a,z)v$ for all $a \in (V^\natural)^{\langle\tau_e\rangle}$ and $v \in V_e^\natural(1/16)$ by definition of the conjugate module. Then we have

$$Y_{V_e^\natural(1/16)^{\sigma_e}}(a,z)\varphi\rho v = \varphi Y_{V^\natural}(\sigma_e a,z)\rho v = \varphi Y_{V^\natural}(\rho a,z)\rho v = \varphi\rho Y_{V^\natural}(a,z)v$$

for any $a \in (V^\natural)^{\langle\tau_e\rangle}$ and $v \in V_e^\natural(1/16)$. Thus $\varphi\rho$ defines a $(V^\natural)^{\langle\tau_e\rangle}$-isomorphism between $V_e^\natural(1/16)$ and $V_e^\natural(1/16)^{\sigma_e}$, which is a contradiction. ∎

Corollary 5.13. *The 2A-orbifold construction applied to the moonshine VOA $V^\natural$ yields $V^\natural$ itself again.*

Proof: Follows from Theorem 5.8 and Corollary 3.15. ∎

Remark 5.14. The statement in the corollary above was conjectured by Tuite [Tu]. In [Tu], Tuite has shown that any $\mathbb{Z}_p$-orbifold construction of $V^\natural$ yields the moonshine VOA $V^\natural$ or the Leech lattice VOA V_Λ under the uniqueness conjecture of the moonshine VOA which states that $V^\natural$ constructed by Frenkel et. al. [FLM] is the unique holomorphic VOA with central charge 24 whose weight one subspace is trivial.

Finally, we end this paper by presenting the modular transformations of characters of $V\!B^0$-modules. Here the character means the conformal character, not the q-dimension, of modules. Recall the characters of $L(1/2,0)$-modules. By our explicit construction in Section 3.1, one can easily prove the following (cf. [FFR] [FRW]):

$$\mathrm{ch}_{L(1/2,0)}(\tau) = (1/2)\cdot q^{-1/48}\left\{\prod_{n=0}^{\infty}(1+q^{n+1/2}) + \prod_{n=0}^{\infty}(1-q^{n+1/2})\right\},$$

$$\mathrm{ch}_{L(1/2,1/2)}(\tau) = (1/2)\cdot q^{-1/48}\left\{\prod_{n=0}^{\infty}(1+q^{n+1/2}) - \prod_{n=0}^{\infty}(1-q^{n+1/2})\right\},$$

$$\mathrm{ch}_{L(1/2,1/16)}(\tau) = q^{-1/24}\prod_{n=1}^{\infty}(1+q^{n}).$$

The following modular transformations are well-known:

$$\begin{aligned}
\mathrm{ch}_{L(1/2,0)}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{L(1/2,0)}(\tau) + \frac{1}{2}\mathrm{ch}_{L(1/2,1/2)}(\tau) + \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,1/16)}(\tau),\\
\mathrm{ch}_{L(1/2,1/2)}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{L(1/2,0)}(\tau) + \frac{1}{2}\mathrm{ch}_{L(1/2,1/2)}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,1/16)}(\tau),\\
\mathrm{ch}_{L(1/2,1/16)}(-1/\tau) &= \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,0)}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{L(1/2,1/2)}(\tau).
\end{aligned}$$

Set $j(\tau) := J(\tau) - 744$, where $J(\tau)$ is the famous $\mathrm{SL}_2(\mathbb{Z})$-invariant. Since $\mathrm{ch}_{V^\natural}(\tau) = j(\tau)$ and

$$\mathrm{ch}_{V^\natural}(\tau) = \mathrm{ch}_{L(1/2,0)}(\tau)\mathrm{ch}_{V\!B^0}(\tau) + \mathrm{ch}_{L(1/2,1/2)}(\tau)\mathrm{ch}_{V\!B^1}(\tau) + \mathrm{ch}_{L(1/2,1/16)}(\tau)\mathrm{ch}_{V\!B_T}(\tau),$$

we can write down the characters of irreducible $V\!B^0$-modules by using those of $V^\natural$ and $L(1/2,0)$-modules. This computation is already done in [Ma] by using Norton's trace formula. The results are written as a rational expression involving the functions $j(\tau)$, $\mathrm{ch}_{L(1/2,h)}(\tau)$, $h = 0, 1/2, 1/16$, their first and second derivatives and the Eisenstein series $E_2(\tau)$ and $E_4(\tau)$, see [Ma].

By Zhu's theorem [Z], the linear space spanned by $\{\mathrm{ch}_{V\!B^0}(\tau), \mathrm{ch}_{V\!B^1}(\tau), \mathrm{ch}_{V\!B_T}(\tau)\}$ affords an $\mathrm{SL}_2(\mathbb{Z})$-action. By using the modular transformations for $j(\tau)$ and $\mathrm{ch}_{L(1/2,h)}(\tau)$, $h = 0, 1/2, 1/16$, we can show the following modular transformations:

$$\begin{aligned}
\mathrm{ch}_{V\!B^0}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{V\!B^0}(\tau) + \frac{1}{2}\mathrm{ch}_{V\!B^1}(\tau) + \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B_T}(\tau),\\
\mathrm{ch}_{V\!B^1}(-1/\tau) &= \frac{1}{2}\mathrm{ch}_{V\!B^0}(\tau) + \frac{1}{2}\mathrm{ch}_{V\!B^1}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B_T}(\tau),\\
\mathrm{ch}_{V\!B_T}(-1/\tau) &= \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B^0}(\tau) - \frac{1}{\sqrt{2}}\mathrm{ch}_{V\!B^1}(\tau).
\end{aligned}$$

Namely, we have exactly the same modular transformation laws for the Ising model $L(1/2,0)$. As in Theorem 5.8, we also note that the fusion algebra for $V\!B^0$ is also canonically isomorphic to that of $L(1/2,0)$. Therefore, we may say that $L(1/2,0)$ and $V\!B^0$ form a dual-pair in the moonshine VOA $V^\natural$.

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